PONTRYAGIN MULTIPLICATION AND A RELATION WITH THE WHITEHEAD PRODUCT

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Pontryagin Multiplication and a Relation with the Whitehead Product

by

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ABSTRACT

The space Ω of loops over a 1-connected space X possesses a natural multiplication, namely the composition of loops. This multiplication induces a Pontryagin product in homology, while the properties of the multiplication induce, in homology, the structure of an associative algebra with identity.

The first three chapters of this thesis are introductory: The first describes the concept of cubic homology, the second, spectral sequences, and the third, the spectral sequence of a fiber space.

In Chapter 4, the Pontryagin product is defined for P, a path space over a 1-connected space X. Since P is well known to be a fiber space over X with fiber Ω, the results of the previous chapters may be used to determine certain properties of the Pontryagin product in P (and Ω); these results are summarized by Theorem (4.45).

Finally in Chapters 5 and 6, the two main theorems are presented and proven in complete detail. Theorem A, due to Bott and Samelson, determines the Pontryagin algebra of the loop space Ω, where the elements of the homology groups of X are transgressive (in the spectral sequence of P), and its corollaries determine the Pontryagin algebra of the loop space over a sphere, the loop space over the one point union of spheres, and the loop space over the suspension of a 0-connected space. Theorem B, due to Samelson, gives a relationship between the Whitehead and the Pontryagin products. Two proofs are given for Theorem B: the first establishes the relationship up to a factor of ± 1, the second determines the sign.

Both Theorem B and the second corollary of Theorem A are valuable in the determination of the homotopy groups of the one point union of spheres.

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INTRODUCTION

One of the most serious disadvantages of homology is the lack of a natural multiplication. When, in 1939, Pontryagin [18] defined a multiplication in the homology groups of a topological group, the multiplication being induced from the product operation in the topological group, he contributed significently to the homology of such spaces. More recently, the Pontryagin product has been extended to a larger class of spaces, namely the H-spaces.

Let X be a 1-connected topological space, let P be the space of paths in X, and let Ω be the space of loops in X, with base point x_o. The space Ω possesses a natural multiplication, namely the composition of loops, which induces a Pontryagin product in its homology groups; if $u \in H_p(\Omega)$ and $v \in H_q(\Omega)$, then $u * v \in H_{p+q}(\Omega)$. The properties of the multiplication in Ω give rise, in homology, to the structure of an associative algebra with identity.

The first three chapters of this paper introduce the basic tools for a consideration of the Pontryagin product in Ω. The first describes the now well known concepts of cubical singular homology, which Eilenberg and MacLane [6] have shown to be identical to the usual singular homology. Chapter 2 introduces the concept of a spectral sequence, especially that of the spectral sequence of a filtered differential group. Chapter 3 defines a fiber space and obtains the homology spectral sequence of a fiber space. The methods used are similar to those of [22], with the modifications of [3] being fully developed.

The Appendix to Chapter 3 shows the importance of the spectral sequence of a fiber space by mentioning several of its applications.

With Chapter 4 begins the main part of the thesis where the Pontryagin product is defined for the space P. Using the fact that P is a fiber space over X with fiber Ω, useful results may be obtained about the Pontryagin product for the space P, including the associativity of the product and the existence of an identity. These results are contained in theorem (4.45).

The first of the two main theorems,

Theorem A. For a given 1-connected space X, we take $H(X)$, with coefficients in the principal ideal domain R, as being R-free and all the elements of $H(X)$ as being transgressive (in P).

Now the Pontryagin algebra $H_*(\Omega)$ is the free associative algebra, with unit, generated by a subgroup of $H(\Omega)$ which is isomorphic to the positive dimensional elements of $H(X)$ under a map reducing dimension by one.,

which is due to [3], is proven in detail in Chapter 5. The corollaries to Theorem A determine the Pontryagin algebras of the loop space of a sphere, of the loop space of the one point union of k spheres,and of the loop space of the suspension of a 0-connected space.

The Whitehead product, originally defined by J. H. C. Whitehead (cf. [30]) in 1941, has been redefined (e.g. in [28] and [29]) and generalized (e.g. in [1] and [19]) many times. In Chapter 6, a relationship is given between the Whitehead and the Pontryagin products.

<u>Theorem B</u>. If, for a 1-connected space X, we have $\alpha \in \pi_{p+1}(X)$, $\beta \in \pi_{q+1}(X)$, $p,q \geq 1$, then

$$\tau[\alpha,\beta] = \pm(\tau\alpha * \tau\beta - (-1)^{pq}\,\tau\beta * \tau\alpha),$$

where the map τ is the composite of a natural isomorphism $\pi_{p+1}(X) \approx \pi_p(\Omega)$ and the Hurewicz homomorphism, the operation [,] is the Whitehead product, and the operation $*$ is the Pontryagin product.,

due to [20], gives this relationship. Two proofs are given for Theorem B : the first leaves the sign open and the second determines the sign to be $(-1)^p$.

Both Theorem B and the second corollary to Theorem A are valuable in computing the homotopy groups of the one point union of k spheres (cf.[9]).

CHAPTER 1

Cubic Homology

The usual definition of singular homology uses the unit n-simplex; however, to develop the spectral sequence of a fiber space, we need an equivalent definition which uses the unit n-cube instead of the unit n-simplex.

We now introduce the basic concepts of the cubic theory.

(1.1) By a singular n-cube in a space X, we mean a map $u : I^n \to X$. If $n = 0$, then u is interpreted as a single point in X. If $n > 0$, we define ith lower and upper faces $\lambda_i^o u$ and $\lambda_i^1 u$ of u to be the singular $(n - 1)$ cubes given by

$(\lambda_i^\varepsilon u)(x_1, \ldots, x_{n-1}) = u(x_1, \ldots, x_{i-1}, \varepsilon, x_i, \ldots, x_{n-1})$,

$i = 1, \ldots, n$; $\varepsilon = 0, 1$; $(x_1, \ldots, x_{n-1}) \in I^{n-1}$. Then, we have

(1.2) $\lambda_i^\varepsilon \lambda_j^\eta = \lambda_{j-1}^\eta \lambda_i^\varepsilon$, $i < j$; $\varepsilon, \eta = 0,1$;

since

$$\begin{aligned}\lambda_i^\varepsilon \lambda_j^\eta u\,(x_1,\ldots,x_{n-2}) &= \lambda_j^\eta u(x_1,\ldots, x_{i-1}, \varepsilon, x_i, \ldots, x_{n-2}) \\ &= u(x_1,\ldots, x_{i-1}, \varepsilon, x_i,\ldots, x_{j-2}, \eta, x_{j-1},\ldots,x_{n-2})\end{aligned}$$

and

$$\begin{aligned}\lambda_{j-1}^\eta \lambda_i^\varepsilon u(x_1,\ldots,x_{n-2}) &= \lambda_i^\varepsilon u(x_1,\ldots, x_{j-2}, \eta, x_{j-1},\ldots, x_{n-2}) \\ &= u(x_1,\ldots,x_{i-1}, \varepsilon, x_i,\ldots,x_{j-2}, \eta, x_{j-1},\ldots,x_{n-2}).\end{aligned}$$

We define $Q_n(X)$ to be the free abelian group generated by all singular n-cubes in X if $n \geq 0$, and $Q_n(X) = 0$ if $n < 0$.

We define an operation d by the formula

$$du = \sum_{i=1}^n (-1)^i (\lambda_i^1 u - \lambda_i^o u).$$

This equation then defines a homomorphism $d : Q_n(X) \to Q_{n-1}(X)$, for every n.

(1.3) Lemma : $\{Q_n(X), d\}$ is a chain complex.

Proof. We need only to show that $dd = 0$.

Take $u \in Q_n(X)$.

Now $ddu = d(\sum_{i=1}^{n}(-1)^i (\lambda_i^1 u - \lambda_i^0 u))$

$$= \sum_{j=1}^{n-1}(-1)^j [\lambda_j^1(\textstyle\sum_{i=1}^n(-1)^i(\lambda_i^1 u - \lambda_i^0 u)) - \lambda_j^0(\sum_{i=1}^n(-1)^i(\lambda_i^1 u - \lambda_i^0 u))]$$

$$= \textstyle\sum_{j=1}^{n-1}(-1)^j[\lambda_j^1(-\lambda_1^1 u + \lambda_1^0 u + \lambda_2^1 u - \lambda_2^0 u - \ldots + (-1)^n(\lambda_n^1 u - \lambda_n^0 u))$$

$$- \lambda_j^0(-\lambda_1^1 u + \lambda_1^0 u + \lambda_2^1 u - \lambda_2^0 u - \ldots + (-1)^n(\lambda_n^1 u - \lambda_n^0 u))]$$

$$= \lambda_1^1\lambda_1^1 u - \lambda_1^1\lambda_1^0 u - \lambda_1^1\lambda_2^1 u + \lambda_1^1\lambda_2^0 u + \ldots - (-1)^n(\lambda_1^1\lambda_n^1 u - \lambda_1^1\lambda_n^0 u)$$

$$- \lambda_1^0\lambda_1^1 u + \lambda_1^0\lambda_1^0 u + \lambda_1^0\lambda_2^1 u - \lambda_1^0\lambda_2^0 u - \ldots + (-1)^n(\lambda_1^0\lambda_n^1 u - \lambda_1^0\lambda_n^0 u)$$

$$- \lambda_2^1\lambda_1^1 u + \lambda_2^1\lambda_1^0 u + \lambda_2^1\lambda_2^1 u - \lambda_2^1\lambda_2^0 u - \ldots + (-1)^n(\lambda_2^1\lambda_n^1 u - \lambda_2^1\lambda_n^0 u)$$

$$+ \lambda_2^0\lambda_1^1 u - \lambda_2^0\lambda_1^0 u - \lambda_2^0\lambda_2^1 u + \lambda_2^0\lambda_2^0 u + \ldots - (-1)^n(\lambda_2^0\lambda_n^1 u - \lambda_2^0\lambda_n^0 u)$$

$$+ \ . \ . \ .$$

$$. \ . \ . \ .$$

$$- (-1)^{n-1}[\lambda_{n-1}^1\lambda_1^1 u - \lambda_{n-1}^1\lambda_1^0 u - \lambda_{n-1}^1\lambda_2^1 u + \lambda_{n-1}^1\lambda_2^0 u + \ldots - (-1)^n(\lambda_{n-1}^1\lambda_n^1 u - \lambda_{n-1}^1\lambda_n^0 u)$$

$$- \lambda_{n-1}^0\lambda_1^1 u + \lambda_{n-1}^0\lambda_1^0 u + \lambda_{n-1}^0\lambda_2^1 u - \lambda_{n-1}^0\lambda_2^0 u + \ldots + (-1)^n(\lambda_{n-1}^0\lambda_n^1 u - \lambda_{n-1}^0\lambda_n^0 u)] .$$

Now for $j < i$, we have $\lambda_j^\varepsilon\lambda_i^\eta = \lambda_{i-1}^\eta\lambda_j^\varepsilon$, $\varepsilon, \eta = 0$ and 1,

so $ddu =$

$$\lambda_1^1\lambda_1^1 u - \lambda_1^1\lambda_1^0 u - \lambda_1^1\lambda_1^1 u + \lambda_1^0\lambda_1^1 u + \ldots - (-1)^n(\lambda_{n-1}^1\lambda_1^1 u - \lambda_{n-1}^0\lambda_1^1 u)$$

$$- \lambda_1^0\lambda_1^1 u + \lambda_1^0\lambda_1^0 u + \lambda_1^1\lambda_1^0 u - \lambda_1^0\lambda_1^0 u - \ldots + (-1)^n(\lambda_{n-1}^1\lambda_1^0 u - \lambda_{n-1}^0\lambda_1^0 u)$$

$$- \lambda_2^1\lambda_1^1 u + \lambda_2^1\lambda_1^0 u + \lambda_2^1\lambda_2^1 u - \lambda_2^1\lambda_2^0 u - \ldots + (-1)^n(\lambda_{n-1}^1\lambda_2^1 u - \lambda_{n-1}^0\lambda_2^1 u)$$

$$+ \lambda_2^0\lambda_1^1 u - \lambda_2^0\lambda_1^0 u - \lambda_2^0\lambda_2^1 u + \lambda_2^0\lambda_2^0 u + \ldots - (-1)^n(\lambda_{n-1}^1\lambda_2^0 u - \lambda_{n-1}^0\lambda_2^0 u)$$

$$+ \ldots$$

...

$$-(-1)^{n-1}[\lambda^1_{n-1}\lambda^1_1 u - \lambda^1_{n-1}\lambda^0_1 u - \lambda^1_{n-1}\lambda^1_2 u + \lambda^1_{n-1}\lambda^0_2 u + \ldots - (-1)^n(\lambda^1_{n-1}\lambda^1_{n-1}u - \lambda^0_{n-1}\lambda^1_{n-1}u$$

$$- \lambda^0_{n-1}\lambda^1_1 u + \lambda^0_{n-1}\lambda^0_1 u + \lambda^0_{n-1}\lambda^1_2 u - \lambda^0_{n-1}\lambda^0_2 u - \ldots + (-1)^n(\lambda^1_{n-1}\lambda^0_{n-1}u - \lambda^0_{n-1}\lambda^0_{n-1}u)].$$

Now it is clearly seen that each term appears twice, once with a positive sign and once with a negative sign; hence all terms cancel and

$$ddu = 0.$$

Thus we have a chain complex, since $Q_n(X)$ is an abelian group and $d : Q_n(X) \to Q_{n-1}(X)$ is such that $dd = 0$.

It would be logical to expect that we need only to pass to homology and we would have the required cubic homology groups of X. However, if we consider the usual definition of a homology theory (as defined by Eilenberg and Steenrod [7], pages 10-12), we have for $X = \{x_o\}$, a one point space, that $H_q(X) = 0$, for $q \neq 0$. For the chain complex $\{Q_n(X), d\}$ such is not the case.

Let us take X to be a single point x_o. For each $q \geq 0$ we have a unique generator for $Q_q(X)$, namely that determined by the unique map $f_q : I^q \to x_o$. Now, if we have $q > 0$ and $i \leq q$, then

$$\lambda^1_i f_q = \lambda^o_i f_q \qquad ,$$

and it follows that f_q is a cycle for each q.

Hence the group of cycles, $Z_q(Q(X))$, is isomorphic to $\mathbb{Z}$, the group of integers, and the group of boundaries, $B_q(Q(X))$, is zero. Thus

$$H_q(Q(X)) \approx \mathbb{Z} \qquad , \quad q \geq 0.$$

We want to eliminate this problem and to do so we introduce the Bott and

Samelson [3] concept of a degenerate cube.

(1.4) Definition: For each integer $P \geq 1$, we denote by $D^{(P)}$ the subgroup generated by those cubes which do not depend on any one of their last P coordinates and put $D_n^{(P)} = D^{(P)} \cap Q_n$. Such cubes are said to be degenerate.

Clearly from this definition we have that

$$D^{(P)} = \sum D_r^{(P)} ,$$

and that $D^{(1)} \subset D^{(2)} \subset \cdots \subset D^{(P)} \subset \cdots$.

If we put $D^{(\infty)} = \bigcup_P D^{(P)}$, then we have that $D_n^{(\infty)} = D_n^{(n)}$.

(1.5) Proposition: $\{D_n^{(P)}(X) , d\}$ forms a normal subcomplex of $\{Q_n(X), d\}$.

Proof. d on $D_n^{(P)}(X)$ is just the restriction of d on $Q_n(X)$ to $D_n^{(P)}(X)$; hence $dd = 0$.

To prove this proposition we need the following Lemma, which has been proven by Bott and Samelson [3].

(1.6) Lemma. (a) For each P, such that $1 \leq P \leq \infty$,

$$d(D^{(P)}) \subset D^{(P)} .$$

(b) The natural homomorphism of the homology group of $(Q/D^{(P)})$ into that of $(Q/D^{(P+1)})$ and that of $(Q/D^{(\infty)})$, induced by $D^{(P)} \subset D^{(P+1)} \subset D^{(\infty)}$, is an isomorphism.

Proof of Lemma: Case 1 : We consider $p < \infty$.

(a) Consider $u \in D^{(P)}$. We apply d and we have $du = \sum_{i=1}^{n} (-1)^i(\lambda_i^1 u - \lambda_i^0 u)$. Now let us assume that u does not depend on some coordinate, say the sth coordinate, with $n-P \leq s$; therefore $\lambda_s^1 u(x_1,\ldots, x_{n-1}) = u(x_1,\ldots,x_{s-1},1,x_s,\ldots,x_{n-1}) = u(x_1,\ldots,x_{s-1},0,x_s,\ldots,x_{n-1})=$

$= \lambda_s^o u(x_1,\ldots,x_{n-1})$ or, in other words, the terms $\lambda_s^o u$ and $\lambda_s^1 u$ cancel.

If $i < s$, $\varepsilon = 0,1$, we have $\lambda_i^\varepsilon u(x_1,\ldots,x_{n-1}) = u(x_1,\ldots,x_{i-1},\varepsilon,x_i,\ldots,$
$\ldots,(x_s),\ldots,x_{n-1})$
and clearly $\lambda_i^\varepsilon u \in D^{(P-1)} \subset D^{(P)}$.

If $i > s$, $\varepsilon = 0,1$, we have $\lambda_i^\varepsilon u(x_1,\ldots,x_{n-1}) =$
$u(x_1,\ldots,(x_s),\ldots,x_{i-1},\varepsilon,x_i,\ldots,x_{n-1})$, and clearly $\lambda_i^\varepsilon u \in D^{(P)}$.

Hence it follows that $d(D^{(P)}) \subset D^{(P)}$.

(b) We consider the exact homology sequence of the triple $(Q, D^{(P+1)}, D^{(P)})$

$$\ldots \xrightarrow{d_*} H_q(D^{(P+1)}, D^{(P)}) \xrightarrow{i_*} H_q(Q,D^{(P)}) \xrightarrow{j_*} H_q(Q,D^{(P+1)}) \xrightarrow{d_*} H_{q-1}(D^{(P+1)},D^{(P)}) \xrightarrow{i_*} \ldots$$

where i_*, j_* are induced by $i: (D^{(P+1)},D^{(P)}) \to (Q,D^{(P)})$, $j : (Q,D^{(P)}) \to (Q,D^{(P+1)})$, respectively, and d_* is the composition of the usual $d : H_q(Q,D^{(P+1)}) \to H_{q-1}(D^{(P+1)})$ and the homomorphism

$H_{q-1}(D^{(P+1)}) \to H_{q-1}(D^{(P+1)},D^{(P)})$ induced by the map $(D^{(P+1)},x_o) \to (D^{(P+1)},D^{(P)})$.

We want to show that the homomorphism j_* is an isomorphism. To do this we use the exactness of the sequence and the fact that
$H_q(D^{(P+1)},D^{(P)}) = 0$ for all q.

To show $H_q(D^{(P+1)},D^{(P)}) = 0$ we consider the linear operator $\Delta : Q_n(X) \to Q_{n+1}(X)$, defined by

$$(1.7)\quad \begin{cases} \Delta u(x_1,\ldots,x_{n+1}) = u(x_2,\ldots,x_n,x_1x_{n+1}), & \text{for } u \in Q_n(X), \quad n > 0, \\ \Delta u(x_1) = u, & \text{for } u \in Q_o(X). \end{cases}$$

We now make the following claim.

(1.8) Claim: Δ has the following properties:

(1) $\Delta(D^{(P)}) \subset D^{(P)}$, for all $P \geq 1$.

(2) Defining [1.] $\tau = 1 + d\Delta + \Delta d$, (1 = identity map), we have $\tau(D^{(P+1)}) \subset D^{(P)}$.

Property (1). If u does not depend on x_r , it is obvious that Δu does not depend on x_{r+1} from the definition; hence, if $u \in D^{(P)}$), $\Delta u \in D^{(P)}$, and Property (1) follows.

Property (2). Let us take $u \in Q_n$ and apply τ.

$$\tau u = 1u + d\Delta u + \Delta du$$

$$\tau u(x_1,\ldots,x_n) = u(x_1,\ldots,x_n) + \textstyle\sum_{i=1}^{n+1}(-1)^i(\lambda_i^1\Delta u(x_1,\ldots,x_n) - \lambda_i^o\Delta u(x_1,\ldots,x_n))$$
$$+ \textstyle\sum_{j=1}^{n}(-1)^j(\Delta\lambda_j^1 u(x_1,\ldots,x_n) - \Delta\lambda_j^o u(x_1,\ldots,x_n))$$

(since Δ is linear).

Applying the operators λ_i^ε , λ_j^ε , ($\varepsilon = 0,1; i = 1,\ldots,n+1;\ j = 1,\ldots, n$), and Δ , we have

$$\tau u(x_1,\ldots,x_n) = u(x_1,\ldots,x_n) - u(x_1,\ldots,x_{n-1}, 1x_n) + u(x_1,\ldots,x_{n-1},0)$$
$$+ u(1,x_2,\ldots,x_{n-1},x_1x_n) - u(0,x_2,\ldots,x_{n-1},x_1x_n)$$
$$- u(x_2,1,x_3,\ldots,x_{n-1},x_1x_n) + u(x_2,0,x_3,\ldots,x_{n-1},x_1x_n)$$
$$+ \ldots$$
$$\ldots$$

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1. This definition of τ differs from than given by Bott and Samelson [3]. The reason for this difference is that they define $du = \sum_i(-1)^i(\lambda_i^o u - \lambda_i^1 u)$, whereas, we define $du = \sum_i(-1)^i(\lambda_i^1 u - \lambda_i^o u)$, and this difference allows to get the same result, namely that $H_q(D^{(P+1)}, D^{(P)}) = 0$.

$$+ (-1)^{n+1}(u(x_2,\dots,x_n,x_1(1)) - u(x_2,\dots,x_n,0))$$
$$- u(1,x_2,\dots,x_{n-1},x_1x_n) + u(0,x_2,\dots,x_{n-1},x_1x_n)$$
$$+ u(x_2,1,x_3,\dots,x_{n-1},x_1x_n) - u(x_2,0,x_3,\dots,x_{n-1},x_1x_n)$$
$$- \dots$$
$$\dots$$
$$+ (-1)^n(u(x_2,\dots,x_{n-1},x_1x_n,1) - u(x_2,\dots,x_{n-1},x_1x_n,0)).$$

Clearly many of the terms appear twice, each time with a different sign, and so cancel; hence we are left with

$$\begin{aligned}\tau u(x_1,\dots,x_n) = {} & u(x_1,\dots,x_{n-1},0) \\ & + (-1)^{n+1}(u(x_2,\dots,x_n,x_1) - u(x_2,\dots,x_n,0)) \\ & + (-1)^n(u(x_2,\dots,x_{n-1},x_1x_n,1) - u(x_2,\dots,x_{n-1},x_1x_n,0)),\end{aligned}$$

or, in other words,

$$\begin{aligned}\tau u = {} & \lambda_1^o\Delta u + (-1)^{n+1}((\lambda_{n+1}^1\Delta)u - (\lambda_{n+1}^o\Delta)u) \\ & + (-1)^n(\Delta\lambda_n^1)u - (\Delta\lambda_n^o)u).\end{aligned}$$

We take $u \in D^{(P+1)}$ and assume that u does not depend on the r^{th} coordinate; then Δu does not depend on the $(r+1)^{st}$ coordinate, and $\lambda_o^1\Delta u$ is also independent of its $(r+1)^{st}$ coordinate. Hence $\lambda_1^o\Delta u \in D^{(P)}$. Similarly, we may show that, for $u \in D^{(P+1)}$, $\lambda_{n+1}^{\varepsilon}\Delta u$ and $\Delta\lambda_n^{\varepsilon}u$ belong to $D^{(P)}$, $\varepsilon = 0,1$. Consequently $\tau u \in D^{(P)}$. Should $P = 0$, i.e. $u \in D^{(1)}$, we have the terms of τu not depending on the value of ε; $\varepsilon = 0,1$; hence all the terms of τu cancel.

We return to the proof of (1.6).

Let x be a cycle of $D^{(P+1)}$, mod $D^{(P)}$ (i.e. $x \in D^{(P+1)}$, $dx \in D^{(P)}$); we have that $d\Delta x = \tau x - \Delta dx - x$ (from the definition of τ), with $\Delta x \in D^{(P+1)}$ (property (1) of (1.8)) and τx and Δdx belonging to $D^{(P)}$ (from (1.8)).

Since τx and $\Delta dx \in D^{(P)}$, we have that $\tau x - \Delta dx$ belongs to the zero class of $D^{(P+1)}$, mod $D^{(P)}$, and we can say $x \sim 0$ in $D^{(P+1)}$, mod $D^{(P)}$; so $H(D^{(P+1)}, D^{(P)})$ is 0.

<u>Case 2:</u> For $P = \infty$.

The assertion for $P = \infty$ now follows from $D^{(\infty)} = \bigcup_P (D^{(P)})$.

Now the proof of proposition (1.5) follows from (1.6).

Using (1.5) we can now consider the various $(Q_n/D_n^{(P)})$ with the boundary d defined as $d[x] = [dx]$, $x \in Q_n$; this leads us to consider the homology groups.

(1.9) <u>Proposition</u>: The identification of the homology groups of the various $(Q_n/D_n^{(P)})$ is natural.

<u>Proof</u>. We want to prove that passing to homology is natural.
i.e. we want the following diagram to be commutative, for $f : X \to Y$ any mapping.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "H"'] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[d, "H"] \\ H\left(\frac{Q(X)}{D^{(P)}(X)}\right) \arrow[r, "f_*"'] & H\left(\frac{Q(Y)}{D^{(P)}(Y)}\right) \end{tikzcd}$$

Let $x \in Q_n(X)$, and let us define a n-dim. cube y of Y by $y = f \circ x$.

Now if x does not depend on some coordinate, say x_r; clearly y will not depend on coordinate x_r. We may define

$$f' : \frac{Q_n(X)}{D_n^{(P)}(X)} \to \frac{Q_n(Y)}{D_n^{(P)}(Y)}$$

by $f'[x] = [y] = [f \circ x]$; f_* is induced by f'.

The proposition now follows by passing to homology and (1.6).

(1.10) We call any of the $Q/D^{(P)}$ the group $C(X)$ of chains of X and its homology group will be called the cubical singular homology group $H(X)$ of X.

For our purposes, we shall use $Q/D^{(\infty)}$; any cube $x \in D^{(\infty)}$ will be called degenerate.

Earlier we saw that the homology groups obtained from the chain complex $\{Q_n(X), d\}$ did not give a homology theory in the Eilenberg-Steenrod sense; we now check to see if the factoring of $Q_n(X)$ by $D_n^{(\infty)}$ corrects this problem.

We again take X to be the single point x_o, and for each $q \geq 0$ we have a unique generator for $Q_q(X)$, determined by $f_q : I^q \to x_o$. Again, for $q \geq 0$, $i \leq q$, $\lambda_i^1 f_q = \lambda_i^o f_q$ and f_q is a cycle for each q. But $f_q : I^q \to x_o$ is a constant map and as such is degenerate for all $q \geq 1$. For $q = 0$, f_q is no longer degenerate.

Hence the group of cycles, $Z_q\left(\dfrac{Q(X)}{D^{(\infty)}(X)}\right)$ is 0, for $q \geq 1$, and is $\mathbb{Z}$ for $q = 0$,

and the group of boundaries, $B_q\left(\dfrac{Q(X)}{D^{(\infty)}(X)}\right)$ is 0, for all q.

Thus

$$H_q\left(\frac{Q(X)}{D^{(\infty)}(X)}\right) \approx \begin{cases} \mathbb{Z}, & \text{for } q = 0 \\ 0, & \text{for } q \geq 1, \end{cases}$$

and the dimension axiom is satisfied.

It would be routine to check that the other conditions of an Eilenberg-Steenrod Homology Theory are satisfied.

Clearly all our cubic chain groups are free, and hence our cubic homology will hold with arbitrary coefficients.

In fact the homology groups given by the cubic chain complex $\{C(X),d\}$ are the same as those given by the usual singular chain complex[2].

Now we know that the results which we have for our usual singular homology theory also hold for our cubic homology theory. Because of this, since our X is defined to be pathconnected, we may restrict ourselves to those cubes all of whose vertices lie at x_o.

In the future we refer to our cubic homology simply as singular homology.

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[2] This is a well known fact and proof of it may be found in [6] , [22], and others.

CHAPTER 2

Spectral Sequences

Before proceeding further we need the use of a tool which was originally developed between 1947 and 1952 by Koszul [15], Leray [17], and Serre[22]; namely the spectral sequence. We shall first define a spectral sequence in the general sense; then, in detail,trace the construction of the Serre spectral sequence.

Our concept of a spectral sequence is the same as that of Spanier [23], page 466.

(2.1) Definition. A bigraded module E (over a principal ideal domain R) is defined to be an indexed collection of R-modules $E_{s,t}$, for all $s,t \in \mathbb{Z}$.

(2.2) Definition. A differential $d : E \to E$, of bidegree $(-r, r-1)$ is defined to be the collection of homomorphism $d : E_{s,t} \to E_{s-r,t+r-1}$, for $s,t \in \mathbb{Z}$, such that $dd = 0$.

(2.3) Definition. The homology module $H(E)$ is defined by

$$H_{s,t}(E) = \frac{\text{Kernal}(d : E_{s,t} \to E_{s-r,t+r-1})}{\text{Image}(d : E_{s+r,t-r+1} \to E_{s,t})}.$$

Using the above definitions we are able to give the Spanier definition of a spectral sequence.

(2.4) Definition. An E^p spectral sequence is a sequence $\{E^r, d^r\}$, $r \geq p$, such that

(1) E^p is a bigraded module and d^r is a differential of bidegree $(-r, r-1)$ on E^r,

(2) for $r \geq p$, there is an isomorphism

$$H(E^r) \underset{\approx}{\to} E^{r+1}.$$

We remark that p is usually 0,1, or 2. We also note that this definition lends itself to the following illustration.

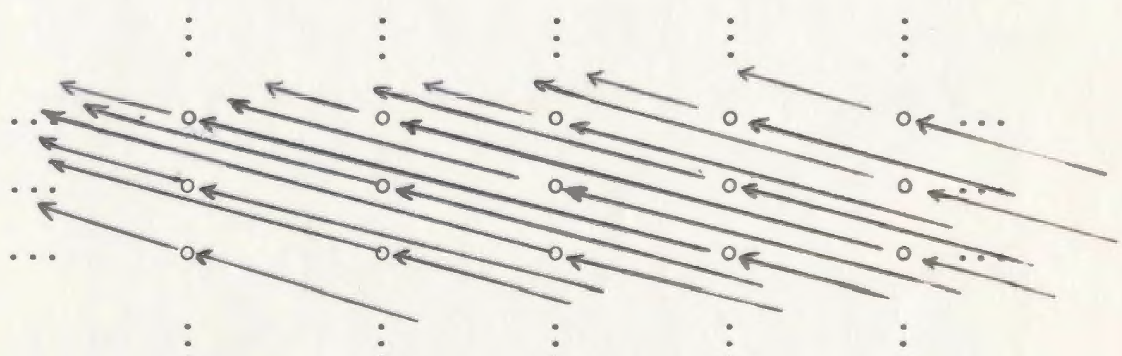
We take $p = 1$.

$$(E^1, d^1) \qquad \begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ \cdots & \longleftarrow & E^1_{0,2} & \longleftarrow & E^1_{1,2} & \longleftarrow & E^1_{2,2} & \longleftarrow & \cdots \\ \cdots & \longleftarrow & E^1_{0,1} & \longleftarrow & E^1_{1,1} & \longleftarrow & E^1_{2,1} & \longleftarrow & \cdots \\ \cdots & \longleftarrow & E^1_{0,0} & \longleftarrow & E^1_{1,0} & \longleftarrow & E^1_{2,0} & \longleftarrow & \cdots \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

$$(E^2, d^2) \qquad \begin{array}{ccccc} & \vdots & \vdots & \vdots & \\ \cdots & E^2_{0,2} & E^2_{1,2} & E^2_{2,2} & \cdots \\ \cdots & E^2_{0,1} & E^2_{1,1} & E^2_{2,1} & \cdots \\ \cdots & E^2_{0,0} & E^2_{1,0} & E^2_{2,0} & \cdots \\ & \vdots & \vdots & \vdots & \end{array}$$

As r increases, we may use points to represent each of the $E^r_{s,t}$.

(E^3, d^3)



and so forth ...

As we are presently only interested in the Serre spectral sequence, we shall not continue to develop this general definition. However, during the development of the Serre spectral sequence we shall meet properties which are just special cases of properties of spectral sequences in general; when we meet such properties we shall mention the general case as a remark.

For the development of the Serre spectral sequence we follow the original Serre paper [22] adding the details where they have been omitted.

(2.5) Definition. Given (A,d) a differential group (i.e. an abelian group A with an endomorphism d such that $dd = 0$), we say that a family of subgroups (A^p) , $p \in \mathbb{Z}$, define on A an increasing filtration if the following conditions are satisfied.

(1) $\bigcup_p A^p = A$

(2) $A^p \subset A^{p+1}$

(3) $d(A^p) \subset A^p$.

The definition may be completed by putting $A^{-\infty} = 0$ and $A^{+\infty} = A$.

For $x \in A$, we let $w(x)$ be the lower bound of all p such that $x \in A^p$. We now claim that the mapping $x \rightsquigarrow w(x)$ satisfies the following properties:

(2.6)
$$w(a - b) \leq \sup (w(a) , w(b)) ,$$
$$w(da) \leq w(a) .$$

The second follows immediately from $d(A^{w(a)}) \subset A^{w(a)}$. In the first case, we take $a \in A^{w(a)}$, $b \in A^{w(b)}$. If $w(a) \leq w(b)$, we have $a \in A^{w(b)}$ since $A^p \subset A^{p+1}$. $A^{w(b)}$ is a subgroup so $(a - b) \in A^{w(b)}$; hence

$w(a - b) \leq w(b)$. Should $w(b) \leq w(a)$ the reasoning is similar; then combining both we have the result.

Conversely, we may use (2.6) to obtain (2.5). If we have given a function w defined on a group A with integral values such that (2.6) is satisfied, we can define an increasing filtration $\{A^p\}$ by taking $A^p = \{a \in A \mid w(a) \leq p\}$. This definition can be shown to satisfy the conditions of (2.5).

We now introduce the following notations

$$Z_p^r = \{x \in A^p \mid d(x) \in A^{p-r}\}$$

(2.7) $$Z_p^\infty = \{x \in A^p \mid d(x) = 0\}$$

$$B_p^r = \{x \in A^p \mid x = dy \ , \ y \in A^{p+r}\}$$

$$B_p^\infty = \{x \in A^p \mid x = dy \ , \ y \in A^q \ , \ q \in \mathbb{Z}\} \ .$$

Clearly from the definition of these subgroups of A^p and the fact that $A^p \subset A^{p+1}$, we have the following inclusions

(2.8) $$B_p^o \subset B_p^1 \subset \cdots \subset B_p^{r-1} \subset B_p^r \subset \cdots \subset B_p^\infty \subset Z_p^\infty \subset \cdots$$
$$\cdots \subset Z_p^r \subset Z_p^{r-1} \subset \cdots \subset Z_p^1 \subset Z_p^o = A^p \ .$$

Since $Z_{p+r}^r = \{x \in A^{p+r} \mid d(x) \in A^p\}$ and $B_p^r = \{x \in A^p \mid x = d(y) \ , \ y \in A^{p+r}\}$, we have

(2.9) $$d(Z_{p+r}^r) = B_p^r \ .$$

(2.10) <u>Definition</u>. We put $E_p^r = Z_p^r/(Z_{p-1}^{r-1} + B_p^{r-1})$.

From (2.8) and (2.9) we have $d(Z_p^r) = B_{p-r}^r \subset \cdots \subset Z_{p-r}^r$; and from (2.9) and the fact that $dd = 0$, we have $d(Z_{p-1}^{r-1} + B_p^{r-1}) = B_{p-r}^{r-1}$; hence our d is compatible with the passing to quotient which defines E_p^r . Now d induces a homomorphism

$$(2.11) \qquad d_p^r : E_p^r \to E_{p-r}^r$$

defined by $d[z] = [dz]$, $z \in Z_p^r$.

We investigate further our induced homomorphism d_p^r .

(2.12) Claim. (i) kernel $d_p^r = (Z_p^{r+1} + Z_{p-1}^{r-1})/(Z_{p-1}^{r-1} + B_p^{r-1})$

(ii) image $d_{p+r}^r = (B_p^r + Z_{p-1}^{r-1})/(Z_{p-1}^{r-1} + B_p^{r-1})$.

Proof of Claim.

(i) We want the kernel of the homomorphism

$$d_p^r : E_p^r \to E_{p-r}^r \quad .$$

Since $E_{p-r}^r = Z_{p-r}^r / (Z_{p-r-1}^{r-1} + B_{p-r}^{r-1})$, an element of E_{p-r}^r is zero if and only if it belongs to

$$\left(\frac{Z_{p-r-1}^{r-1} + B_{p-r}^{r-1}}{Z_{p-r-1}^{r-1} + B_{p-r}^{r-1}} \right).$$

Now $d_p^r\,(E_p^r) = d_p^r \left(\dfrac{Z_p^r}{Z_{p-1}^{r-1} + B_p^{r-1}} \right) = \dfrac{d_p^r\,(Z_p^r)}{Z_{p-r-1}^{r-1} + B_{p-r}^{r-1}}$, by (2.11), and

an element of $d_p^r(E_p^r)$ is zero if and only if it belongs to

$\left(\dfrac{d_p^r(Z_p^r) \cap (Z_{p-r-1}^{r-1} + B_{p-r}^{r-1})}{d_p^r(Z_p^r) \cap (Z_{p-r-1}^{r-1} + B_{p-r}^{r-1})} \right)$ i.e. if and only if it belongs to

$\left(\dfrac{B_{p-r}^r \cap (Z_{p-r-1}^{r-1} + B_{p-r}^{r-1})}{B_{p-r}^r \cap (Z_{p-r-1}^{r-1} + B_{p-r}^{r-1})} \right)$, since $d_p^r(Z_p^r) = B_{p-r}^r$ by (2.9).

Now $B^r_{p-r} \cap Z^{r-1}_{p-r-1} = B^{r+1}_{p-r-1}$, by applying the definition, and $B^r_{p-r} \cap B^{r-1}_{p-r} = B^{r-1}_{p-r}$, since $B^{r-1}_{p-r} \subset B^r_{p-r}$ by (2.8).

Hence $d^r_p(\ker d^r_p) = \dfrac{B^{r+1}_{p-r-1} + B^{r-1}_{p-r}}{B^{r+1}_{p-r-1} + B^{r-1}_{p-r}} \subset E^r_{p-r}$

i.e. $d^r_p(\ker d^r_p) = \dfrac{d^r_p(Z^{r+1}_p + Z^{r-1}_{p-1})}{B^{r+1}_{p-r-1} + B^{r-1}_{p-r}}$ by (2.9),

i.e. $\ker d^r_p = \dfrac{Z^{r+1}_p + Z^{r-1}_{p-1}}{Z^{r-1}_{p-1} + B^{r-1}_p}$ by (2.11).

(ii) The proof of (ii) follows immediately along similar lines.

Applying (2.8) to (2.12) we see that $\operatorname{Im} d^r_{p+r} \subset \ker d^r_p$ which implies that $d^r_p \circ d^r_{p+r} = 0$ and we are able to consider the homology groups.

(2.13) <u>Lemma</u>. $H(E^r_p) = E^{r+1}_p$

<u>Proof</u>. We consider the quotient of the kernal of d^r_p and the image of d^r_{p+r}.

$$\frac{\ker d^r_p}{\operatorname{Im} d^r_{p+r}} = \frac{\dfrac{Z^{r+1}_p + Z^{r-1}_{p-1}}{Z^{r-1}_{p-1} + B^{r-1}_p}}{\dfrac{B^r_p + Z^{r-1}_{p-1}}{Z^{r-1}_{p-1} + B^{r-1}_p}} \quad \text{by (2.12)}$$

$$\approx \frac{Z^{r+1}_p + Z^{r-1}_{p-1}}{Z^{r-1}_{p-1} + B^r_p} \quad \text{by an algebraic property of quotients.}$$

$$= \frac{Z^{r+1}_p}{(Z^{r-1}_{p-1} + B^r_p) \cap Z^{r+1}_p}$$

since if an element belongs to Z^{r-1}_{p-1} it also belongs to the 0-class.

$$= \frac{Z_p^{r+1}}{Z_{p-1}^r + B_p^r}$$ since $B_p^r \subset Z_p^{r+1}$ by (2.8) and $Z_p^{r+1} \cap Z_{p-1}^{r-1} = Z_{p-1}^r$ by applying the definitions.

$$= E_p^{r+1}$$ by (2.10).

i.e. $H(E_p^r) \approx E_p^{r+1}$.

From these results we obtain our spectral sequence.

(2.14) Definition. We put $E^r = \Sigma_p E_p^r$. The summands E_p^r define on E^r a graded structure. The elements of E_p^r are said to be of filtered degree p ; the homomorphisms d_p^r define on E^r a homogeneous differential d^r of degree-r with respect to the filtered degree. The sequence of graded differential groups (E^r) , $r = 0,1, \ldots$, is the Serre spectral sequence or the spectral sequence attached to the filtered differential group A.

Now (2.13) tells us that the homology group of E^r , calculated in E_p^r, is isomorphic to E_p^{r+1} , and we have $H(E^o) = E^1$; $H(E^1) = E^2$; ... ; $H(E^r) = E^{r+1}$; ... .

We now look at the beginning and ending terms of the spectral sequence.

E^o : By definition (2.10) we have

$$E_p^o = Z_p^o / (Z_{p-1}^{-1} + B_p^{-1})$$

$$= \frac{\{x \in A^p | dx \in A^p\}}{\{x \in A^{p-1} | dx \in A^p\} + \{x \in A^p | x = dy, y \in A^{p-1}\}}$$ by (2.7)

$$= \frac{A^p}{A^{p-1}}$$ from (2.5).

(2.15). This tells us that E^o is the direct sum of successive quotients A^p/A^{p-1} and we call it the graded group associated with the filtered group A.

From (2.11) we see that d^o maps E^o_p into itself and that it is obtained from the differential d of A by passage to quotient.

The next stage would naturally be $E^1 = H(E^o) = H(A^p/A^{p-1})$.

$\underline{E^\infty\ :}$

(2.16) Definition. We define the terminal group of the spectral sequence to be the term $E^\infty = \Sigma_p E^\infty_p$, defined by putting $E^\infty_p = Z^\infty_p/(Z^\infty_{p-1} + B^\infty_p)$.

This definition clearly corresponds with the definition of the term E^r and we may interpret it either as the limit of the terms E^r (in a way to be defined later), or as closely bound to $H(A)$. Hence it forms a transition between the E^r and $H(A)$.

We now give a more precise definition of the connection between E^∞ and $H(A)$.

(2.17) Definition. We let D_p be the image of $H(A^p)$ in $H(A)$ induced by the inclusion mapping of A^p into A.

This definition immediately gives us that

(2.18) $$D_p = Z^\infty_p/B^\infty_p$$

and we make the following claim

(2.19) Claim: $$D_p/D_{p-1} = E^\infty_p$$

Proof of Claim: $$D_p/D_{p-1} = \frac{(Z^\infty_p/B^\infty_p)}{(Z^\infty_{p-1}/B^\infty_{p-1})} \quad \text{by (2.18)}$$

$$= Z^\infty_p \,/\, (Z^\infty_{p-1} + B^\infty_p) \quad \text{by (2.7) and (2.8)}$$

$$= E^\infty_p \quad \text{by (2.16).}$$

Now (2.19) implies that if we consider $H(A)$ as filtered by the D_p, the group E^∞ is only the graded group associated with the filtered group $H(A)$.

Remark: The concept of a terminal group also holds for spectral sequences in the general sense, although it is usually called the limit of the spectral sequence; see Spanier [23], page 467.

We now consider the case when our filtered differential group A is also a graded group. Here the results will be parallel to the ones for the ungraded case; hence we number the results with the same number as in the ungraded case and affix a prime. This holds true even if the result is not explicitly stated below.

(2.20) Definition. We assume that A is graded, i.e. A is the direct sum of subgroups ${}^nA$, $n \in \mathbb{Z}$, (we write ${}^nA$ instead of the more usual A^n or A_n so that we can avoid confusion with the filtered groups A^p); we will assume that d is of degree -1 with respect to the graduation (i.e. $d({}^nA) \subset {}^{n-1}A$) and that the filtration is compatible with the graduation (i.e. $A^p = \Sigma_n(A^p \cap {}^nA)$). We put

$$A^{p,q} = {}^{p+q}A \cap A^p$$

and we denote by $H_n(A)$ the nth homology group of A.

We now wish to graduate the terms of the spectral sequence. The existence of a graduation on A allows us to define a graduation on those groups which we defined before, and we have

$$(2.7)' \quad \begin{cases} Z^r_{p,q} = \{x \in A^{p,q} \mid d(x) \in A^{p-r,q+r-1}\} \\ B^r_{p,q} = \{x \in A^{p,q} \mid x = d(y) \ , \ y \in A^{p+r,q-r+1}\} \\ Z^\infty_{p,q} = \{x \in A^{p,q} \mid d(x) = 0\} \\ B^\infty_{p,q} = \{x \in A^{p,q} \mid x = d(y) \ , \ y \in A^{s,t} \ , \quad s,t \in \mathbb{Z}\} \end{cases}$$

(2.17)' $\quad D_{p,q}$ = image of $H(A^{p,q})$ in $H(A)$ induced by the inclusion mapping of $A^{p,q}$ in A,

and

(2.18)' $$D_{p,q} = Z^{\infty}_{p,q}/B^{\infty}_{p,q} \quad .$$

We note that $Z^r_{p,q}$, $B^r_{p,q}$, $B^{\infty}_{p,q}$, $Z^{\infty}_{p,q}$, $D_{p,q}$ are the subgroups of Z^r_p , B^r_p , B^{∞}_p , Z^{∞}_p , D_p, respectively, formed by the homogeneous elements of degree $p + q$. Each of Z^r_p , B^r_p , B^{∞}_p , Z^{∞}_p , D_p is the direct sum of $Z^r_{p,q}$, $B^r_{p,q}$, $B^{\infty}_{p,q}$, $Z^{\infty}_{p,q}$, $D_{p,q}$, respectively, for $-\infty < q < \infty$. We demonstrate for the case of Z^r_p and $Z^r_{p,q}$. By (2.7)', $Z^r_{p,q} = \{x \in A^{p,q} | d(x) \in A^{p-r,q+r-1}\}$, and for $-\infty < q < \infty$, we consider

$$\begin{aligned} \Sigma_q Z^r_{p,q} &= \Sigma_q \{x \in A^{p,q} | d(x) \in A^{p-r,q+r-1}\} \\ &= \{x \in \Sigma_q A^{p,q} | d(x) \in \Sigma_q A^{p-r,q+r-1}\} \\ &= \{x \in A^p | d(x) \in A^{p-r}\} \qquad \text{since graduation is compatible with filtration in } A. \\ &= Z^r_p \quad \text{by (2.7).} \end{aligned}$$

As in the ungraded case we have the corresponding inclusions (which we number (2.8)') and see that

(2.9)' $$d(Z^r_{p+r,q-r+1}) = B^r_{p,q} \quad .$$

(2.10)' <u>Definition.</u> We put $E^r_{p,q} = Z^r_{p,q}/(Z^{r-1}_{p-1,q+1} + B^{r-1}_{p,q})$, for $0 \leq r \leq \infty$.

Now the $E^r_{p,q}$ grade E^r_p , and the corresponding results follow; the term E^r of the spectral sequence is bigraded by the $E^r_{p,q}$; where p is the filtered degree; q the complementary degree; and $(p + q)$ the total degree ($(p + q)$ corresponds to the degree of A). As before we have a differential d^r which now reduces filtered degree by r, total degree by 1, and increases complementary degree by $r - 1$.

(2.14)' Our sequence of bigraded differential groups (E^r), $r = 0,1, \ldots,$ is the Serre spectral sequence (for graded filtered differential groups) or the spectral sequence attached to the graded filtered differential group A.

(2.21) Theorem: The Serre spectral sequence just defined is a well defined spectral sequence.

Proof. Clearly this is an E^o spectral sequence; (2.4) follows immediately from (2.10)', (2.11)', and (2.13)'.

We now consider Serre's "hypothèse supplémentaire", [22], page 431; but we define it as the condition for a filtration to be regular, see, for example, Hu [12], page 236.

(2.22) Definition. A filtration on A is said to be regular if $x \neq 0$ is a homogeneous element of A, then $0 \leq w(x) \leq \deg(x)$.

This definition just states that the filtration and the degree are non-negative and the weight does not exceed the degree. If we go back to (2.20), the definition of $A^{p,q}$, we see that the regularity condition may be expressed as

(2.23) $$A^{p,o} = {}^pA \quad ; \quad A^{p,q} = 0 \quad \text{if} \quad p < 0 .$$

We now introduce the concept of a first-quadrant spectral sequence, see [23], page 468.

(2.24) <u>Definition</u>. A first quadrant spectral sequence is a spectral sequence E having the property that $E^r_{p,q} = 0$ if $p < 0$ or $q < 0$, for $0 \leq r \leq \infty$.

We now have the following lemma.

(2.25) <u>Lemma</u>. If the filtration on A is regular then the associated spectral sequence E is a first quadrant spectral sequence.

<u>Proof</u>. Firstly we want $E^o_{p,q} = 0$ if $p < 0$ or $q < 0$. Now (2.23) gives that, for $p < 0$, $A^{p,q} = 0$; which by (2.7)' implies that $Z^o_{p,q} = 0$.

If $q < 0$, (2.20) implies that $w(x) \geq \deg(x)$, for $p \geq 0$; hence $A^{p,q}$ must be trivial, and again (2.7)' implies $Z^o_{p,q} = 0$.

By (2.10)', $E^o_{p,q} = Z^o_{p,q}/(Z^{-1}_{p-1,q+1} + B^{-1}_{p,q})$; thus $E^o_{p,q} = 0$, if $p < 0$ or $q < 0$.

Now (2.13)' gives us $E^{r+1}_{p,q} = H(E^r_{p,q})$, so if $E^r_{p,q} = 0$, then $E^{r+1}_{p,q} = 0$; hence we can say that $E^r_{p,q} = 0$, if $p < 0$ and $q < 0$.

Clearly it may also be shown that $E^\infty_{p,q} = 0$, if $p < 0$ or $q < 0$.

By (2.19)', $E^\infty_{p,q} = D_{p,q}/D_{p-1,q+1}$, and by (2.25), $E^\infty_{-1,n+1} = 0$; hence we have $D_{-1,n+1} = 0$. We know $D_{n,0} = Z^\infty_{n,0}/B^\infty_{n,0}$ by (2.18)' and hence equal to $H_n(A)$.

(2.26) Thus we have the composition sequence

$$0 = D_{-1,n+1} \subset D_{0,n} \subset \cdots \subset D_{n-1,1} \subset D_{n,0} = H_n(A) \quad \text{and clearly}$$

$$\bigcap_p D_p = 0.$$

(2.27) <u>Proposition</u>. If the filtration on A is regular, then

$$E^r_{p,q} = E^{r+1}_{p,q} = \cdots = E^\infty_{p,q} \quad , \quad \text{for } r > \sup(p,q+1).$$

Proof. If $r > p$, all elements of $E^r_{p,q}$ are cycles for d^r, since d^r reduces the filtered degree by r from (2.14)'; and $E^r_{p-r,q+r-1} = 0$ by (2.25), since $p - r < 0$.

If $r > q + 1$, no non-zero element of $E^r_{p,q}$ can be a boundary for d^r, since d^r increases the complementary degree by $r - 1$, from (2.14)', and $E^r_{p+r,q-r+1} = 0$ by (2.25), since $q - r + 1 < 0$.

Now for $r > \sup(p,q + 1)$ we have $E^r_{p,q} = E^{r+1}_{p,q} = \ldots$ since $H(E^r_{p,q}) = E^{r+1}_{p,q}, \ldots$ by (2.13)'.

For the case of $E^\infty_{p,q}$, it is sufficient to say that for r large enough we have $Z^r_{p,q} = Z^\infty_{p,q}$ and $B^r_{p,q} = B^\infty_{p,q}$.

Going back to what we said previously, this proposition allows us to see in what sense we can say that the group E^∞ is the limit of the groups E^r : for a total degree n given, there exists an r large enough, so that the groups formed by the terms of total degree n of E^r and E^∞ are isomorphic.

(2.28) Remarks (1) We may introduce the concept of convergence of a spectral sequence. Following Spanier [23], page 467, we have the following definitions.

definition(1): A spectral sequence E is said to converge if for every p and q there exists an integer $r(p,q)$, such that for $r > r(p,q)$, $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ is trivial.

definition(2): A spectral sequence E is said to converge in the strong sense if for p and q given, there exists an $r(p,q)$ such that for $r > r(p,q)$, $E^r_{p,q} \approx E^{r+1}_{p,q} \approx \ldots \approx E^\infty_{p,q}$.

Hence our condition in (2.27) is simply convergence in the strong sense.

(2) Spanier [23], page 468, gives that a first quadrant spectral sequence is convergent in the strong sense; therefore, we can now give a trivial proof for (2.27).

CHAPTER 3

The Homology Spectral Sequence of a Fiber Space

We now consider the concept of a fiber space, which, as we shall see in Chapter 4, is an essential tool in the consideration of the Pontryagin multiplication in loop spaces. Actually, fiber spaces are essential for much more than this; they are useful in the axiomatization of homotopy, the computation of homotopy groups, and the application of homotopy to problems in geometry.

Here we want to establish the homology spectral sequence for a fiber space. To do this we follow the approach of Serre [22], developing fully the suggested modifications of Bott and Samelson [3].

Before defining a fiber space we require the following concepts.

(3.1) Consider a map $p : P \to B$ of a space P onto a space B. For X a given space, $f : X \to B$ a given map, and $f_t : X \to B$, $t \in I$, a given homotopy of f, we say that a map $f' : X \to P$ covers f (relative to p) if $pf' = f$, and that a homotopy $f'_t : X \to P$ covers f_t (relative to p) if $pf'_t = f_t$, $t \in I$; f'_t is said to be a covering homotopy of f_t. The map $p : P \to B$ has the covering homotopy property (CHP) for X if, for every map $f' : X \to P$ and every homotopy $f_t : X \to B$, $t \in I$, of the map $f = pf' : X \to B$, there exists a homotopy $f'_t : X \to P$, $t \in I$, of f', such that f'_t covers f_t. The map $p : P \to B$ has the absolute covering homotopy property (ACHP) if it has the CHP for every space X. The map $p : P \to B$ has the polyhedral covering homotopy property (PCHP) if it has the CHP for every polyhedra X.

(3.2) Definition. A fiber space is defined to be the triple (P,p,B), where P and B are spaces and $p : P \to B$ is a mapping of P onto B such that p has the PCHP.

A fiber space defined in this manner is often called a Serre fiber space; we note that the PCHP corresponds to Serre's condition R, [22], page 443. This is not the only definition for a structure of this nature; we have the Hurewicz fiber space, where the ACHP replaces the PCHP, and the concept of a locally trivial fiber space or fiber bundle, which is defined in a slightly different manner. Both the Hurewicz and the locally trivial fiber space contain the Serre fiber space in most cases[3].

We now give the following Proposition which will be useful later.

(3.3) Proposition. Let (P,p,B) be a fiber space, A and X be two finite contractible polyhedra, $A \subset X$, then, if $f : X \to B$ and $g' : A \to P$ are maps such that $p \cdot g' = f|A$, we have that g' has an extension $f' : X \to P$ such that $pf' = f$.

We do not give the proof of this proposition; the proof may be found in [22], page 443, or may be obtained from Theorem 3.1 on page 63 of [12].

(3.4) Definition. For $x \in P$, $b = p(x) \in B$, we call $F = p^{-1}(b)$ the fiber.

Henceforth, we assume that B and F are arcwise connected. Clearly it follows that P is arcwise connected. This assumption allows us to use exclusively those cubes with vertices at x(or b) without changing any of our homology groups. (See end of Chapter 1).

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[3] For more details and definitions see either of [22], [12], [23], or others.

We will now examine the effect of π_1 (B) upon the homology groups of F.

(3.5) Definition. If for a fiber space (P,p,B) and a loop v on B, with endpoints at b, there is a mapping C which assigns to to an n-dimensional singular cube u of F an (n + 1)-dimensional singular cube C(u) of P, it is called a construction for v when the following conditions are satisfied.

(1) $\lambda_1^o C(u) = u$,

(2) $(p \cdot C(u))(t,t_1,\ldots,t_n) = v(t)$,

(3) $C(\lambda_i^\varepsilon u) = \lambda_{i+1}^\varepsilon C(u)$, $\varepsilon = 0,1$,

(4) if $u \in D^{(\infty)}$, $C(u) \in D^{(\infty)}$

We denote by S_c the endomorphism of cubic chains of F defined by $S_c u\,(t_1,\ldots,t_n) = C(u)(1,t_1,\ldots,t_n)$.

From condition (4) $S_c u$ is clearly degenerate whenever u is degenerate and, since $S_c(\lambda_i^\varepsilon u)(t_1,\ldots,t_n) = C(\lambda_i^\varepsilon u)(1,t_1,\ldots,t_n) = \lambda_{i+1}^\varepsilon C(u)(1,t_1,\ldots,t_n) = \lambda_i^\varepsilon(S_c u)(t_1,\ldots,t_n)$ from the definition, we can see that S_c commutes with the boundary.

(3.6) Lemma. For each loop v, there exists at least one construction for v. If v_1, and v_2 are two homotopic loops, and if C_1 and C_2 are the constructions for v_1 and v_2, respectively, then S_{c_1} is homotopically equivalent to S_{c_2}.

We will delay the proof of this lemma until we have introduced certain constructions later in the chapter.

We take a loop v on B of homotopy class $\alpha \in \pi_1(B)$, a construction C for v, and S_c, the endomorphism defined by C. Now S_c defines an endomorphism of the homotopy groups of F; we denote this endomorphism by

T_α as it can only depend on α from (3.6).

(3.7) Proposition. The mapping $\alpha \to T_\alpha$ is a representation of $\pi_1(B)$ in the automorphism group of $H(F)$.

Proof: We know that $\pi_1(B)$ is equipped with group properties obtained from the composition of loops, denoted by $\circ$, with e as the constant loop[4] .

We need only show that $T_e = 1$ and $T_\alpha \circ T_\beta = T_{\alpha\bullet\beta}$.

$T_e = 1$: Consider the cube $Cu(t,t_1,\ldots,t_n) = u(t_1,\ldots,t_n)$. In view of condition (2) of (3.5), Cu is clearly a construction for the constant loop at $b \in B$, since $p \circ Cu\,(t,t_1,\ldots,t_n) = p \circ u\,(t_1,\ldots,t_n) = b$.

Now $S_c u(t_1,\ldots,t_n) = Cu(t,t_1,\ldots,t_n) = u(t_1,\ldots,t_n)$, for all u, so that $T_e = 1$.

$T_\alpha \circ T_\beta = T_{\alpha\bullet\beta}$: For $v \in \alpha$, $v' \in \beta$ we have $v \bullet v' \in \alpha \bullet \beta$; we take C and C' as constructions for v and v' respectively. We now define a construction C'' in the following manner:

$$C''u(t,t_1,\ldots,t_n) = \begin{cases} C'u(2t,t_1,\ldots,t_n) & , \quad 0 \leq t \leq \frac{1}{2} \\ C(S_{c'}u)(2t-1,t_1,\ldots,t_n) & , \quad \frac{1}{2} \leq t \leq 1 . \end{cases}$$

Clearly C'' is a construction for $v'' = v \bullet v'$ by definition. Also $S_{c''}u(t_1,\ldots,t_n) = C''u(1,t_1,\ldots,t_n) = C(S_{c'}u)(1,t_1,\ldots,t_n) = S_c(S_{c'}u)(t_1,\ldots,t_n) = (S_c \circ S_{c'}u)(t_1,\ldots,t_n)$; that is $S_{c''} = S_c \circ S_{c'}$ and by passing to homology groups $T_\alpha \circ T_\beta = T_{\alpha\bullet\beta}$, which proves the proposition.

Hence we have shown that $\pi_1(B)$ operates on $H(F)$ and $H(F)$ forms a local system on B.

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[4] Composition of loops is defined in detail in Chapter 4.

We also mention the following result, proven in [22], page 445.

(3.8) If (P,p,B) is a fiber bundle with an arcwise connected structure group, then $\pi_1(B)$ operates trivially on $H(F)$.

The Filtration of $C(P)$.

We now filter the complex $A = C(P)$, the singular cubic complex of P, and obtain a spectral sequence which we shall use to obtain a relation between the homology of P and that of B and F. We again follow the Serre approach [22] with the adaptions of [3].

From Chapter 1 we see that to filter $A = C(P)$, we need only filter $Q(P)$ by subgroups $\ldots \subset T^r \subset T^{r+1} \subset \ldots$, and take the A^r as the image of these subgroups T^r in A. We now define $T^{r,q} \subset Q_{r+q}(P)$.

(3.9) <u>Definition</u>. $T^{r,q}$ is defined to be the subgroup of $Q(P)$ generated by the $(r + q)$-dimensional cubes u of P such that the cube $p \circ u$, the projection of u on B by p, does not depend on the last q coordinates. We put $T^r = \sum_q T^{r,q}$.

Now by (3.9) we can say that a cube $u \in Q(P)$ is of filtration $\leq r$, if the cube $p \circ u \in Q(B)$ depends only on the first r coordinates, and hence, $T^r \subset T^{r+1}$. Clearly this filtration defined by T^r is regular, for if (2.22) were not satisfied (3.9) would be meaningless.

(3.10). Now if $u \in T^r$, we also have $\lambda_i^\varepsilon u \in T^r$, for all i, and $\lambda_i^\varepsilon u \in T^{r-1} \subset T^r$, if $i \leq r$.

(3.10) follows from the fact that $p \circ u$ depends only on the first r coordinates. Consequently, for $i > r$, $p(\lambda_i^\varepsilon u)$ still depends only on the first r coordinates, and for $i \leq r$, $p(\lambda_i^\varepsilon u)$ can only depend on the first $r-1$ coordinates, from (1.1).

Now (3.10) implies that the T^r are stable for the boundary operator and our A^r, defined as

(3.11) $$A^r = T^r(P)/(T^r(P) \cap D^{(\infty)}) ,$$

is a suitable filtration of $C(P)$. The results of Chapter 2 imply that we now have a Serre spectral sequence for P; we shall directly determine the terms E^0, E^1, E^2 and the differentials d^0, d^1.

Term E^0: From (2.14) and (2.15) we have $E_r^0 = A^r/A^{r-1}$ and $E^0 = \Sigma_r E_r^0$. d^0 is obtained from the boundary operator on A^r by passing to the quotient. Hence by (3.11), E_r^0 is isomorphic to the group generated by linear combinations of cubes u, such that $w(u) \leq r$, modulo the linear combinations of degenerate cubes and cubes such that $w(u) \leq r-1$. Now if u is a cube such that $w(u) \leq r$, we have

(3.12) $$d^0 u = \sum_{i>r} (-1)^i (\lambda_i^1 u - \lambda_i^0 u) \quad \text{in } E_r^0 ,$$

since $\lambda_i^\varepsilon u \in T^{r-1}$ whenever $i \leq r$, from (3.10).

(3.13) Definition. For a cube $u \in T^{r,q}$, we define two operations B and F in the following manner:

$$Bu(t_1,\ldots,t_r) = p \circ u(t_1,\ldots,t_r, Y_1,\ldots,Y_q) \;, \quad Y_i \text{ anything}$$
$$Fu(t_1,\ldots,t_q) = u(0,\ldots,0, t_1,\ldots,t_q).$$

(3.14) Lemma. Bu and Fu have the following properties:

(1) Bu is an r-dimensional cube of B and Fu is an q-dimensional cube of F

(2) if $w(u) \leq r - 1$, Bu is degenerate

(3) if u is degenerate, then either Bu or Fu is degenerate

(4) $B\lambda_i^\varepsilon u = Bu$, $F\lambda_i^\varepsilon u = \lambda_{i-r}^\varepsilon Fu$, if $i > r$, $\varepsilon = 0,1$.

Proof. (1) is obvious.

(2): if $w(u) \leq r - 1$, $p \circ u$ can not depend on all the first r-coordinates and, by (3.13), neither can Bu.

(3): if u does not depend on any of the first r coordinates; clearly, neither can $p \circ u$, so $Bu \in D^{(\infty)}$; if u does not depend on any of the last q coordinates, Fu must be degenerate by (3.13).

(4) follows immediately from (3.13) and (1.1).

(3.15) Definition. We put $J_r = C_r(B) \otimes C(F)$ and define a differential d_F by $d_F(b \otimes f) = (-1)^r b \otimes df$.

(3.16) Definition. We define a homomorphism $\phi : E_r^o \to J_r$ by

$$\phi(u) = Bu \otimes Fu.$$

Because of (3.14), (3.16) is compatible with the definition of E_r^o and ϕ commutes with the differentials.

The term E^1: Using the following Proposition, we will be able to calculate E^1.

(3.17) Proposition. The homomorphism $\phi : E_r^o \to J_r$, defined by (3.16), is a chain equivalence.

Proof. The proof of this proposition requires us to construct a homomorphism $\Psi : J_r \to E_r^o$, such that $\phi \circ \Psi = 1$ and $\Psi \circ \phi = h$, h being a homotopy operator.

(3.18) Lemma. To each pair of cubes (u,v), u a r-dimensional cube of B and v a q-dimensional cube of F, we may associate a cube $W = K(u,v)$ situated in P, of degree $n = r + q$, of filtration $\leq r$, and such that

(1) $B \circ K(u,v) = u \quad , \quad F \circ K(u,v) = v$,

(2) for all $i \leq q$, we have $K(u,\lambda_i^{\varepsilon} v) = \lambda_{i+r}^{\varepsilon} K(u,v)$, $\varepsilon = 0,1$,

(3) if u or v degenerate, then $K(u,v)$ degenerate.

Proof. The proof will be by induction on the integer q.

Part 1 : $q = 0$

Since there is only one 0-cube, the cube v is reduced to the point x, and our problem is, given a map $u : I^r \to B$, which sends all the vertices of I^r into $b \in B$, we have to find a map $w : I^r \to P$, which sends all the vertices of I^r into $x \in P$, such that $p \circ w = u$.

We put $X = I^r$ and $A = \{\omega\}$. [In future, ω will denote the point $(0,\ldots,0)$]. Now applying (3.3) to the pair (X,A), we have a map $w' : I^r \to P$ such that $p \circ w' = u$ and $w'(\omega) = x$. We let s_α be the vertices of I^r and put $f_\alpha = w'(s_\alpha)$. We have $f_\alpha \in F$; since F is assumed arcwise connected, there exist maps $g_\alpha : I \to F$ such that $g_\alpha(0) = f_\alpha$ and $g_\alpha(1) = x$. We use these paths to deform the cube w' into a cube w with the same projection and whose vertices are all at x.

Now we put $X = I^r \times I$ and $A = I^r \times \{0\} \cup \{s_\alpha\} \times I$. Both X and A are obviously contractible. We now define two maps, $f : I^r \times I \to B$ by

$$f(x_1,\ldots,x_r,t) = u(x_1,\ldots,x_r) ,$$

and $g : A \to P$ by

$$g(x_1,\ldots,x_r,t) = \begin{cases} w'(x_1,\ldots,x_r) & , \text{ if } (x_1,\ldots,x_r,t) \in I^r \times \{0\} \\ g_\alpha(t) & , \text{ if } (x_1,\ldots,x_r,t) \in \{s_\alpha\} \times I \end{cases}$$

g is well defined since

$$g(s_\alpha,0) = g_\alpha(0) = f_\alpha = w'(s_\alpha).$$

Applying (3.3) we have a map $h : I^r \times I \to P$ extending g and such that $p \circ h = f$. We define $w : I^r \to P$ by

$$w(y) = h(y,1),\ y \in I^r .$$

Since h extends g, h clearly satisfies conditions (1) and (2). Should u be degenerate (v can not be degenerate as it is a zero cube), we clearly have that w' is degenerate and hence $g : A \to P$ is degenerate. We now collapse A along those coordinates on which g does not depend and extend g to X as before. Clearly our $h : I^r \times I \to P$ also does not depend on those coordinates along which we have collapsed A, for otherwise h would contradict the fact that $h|A = g$.

Since $w(y) = h(y,1)$, w is also degenerate.

Part 2 : from $q - 1$ to q

We take $q \geq 1$ and assume, for $q' < q$, that we have constructed a function $K(u,v)$ satisfying our three conditions. We now construct K when v is of dimension q.

We transform the problem of constructing our cube $w = K(u,v)$ into the problem of covering a mapping.

We put $X = I^r \times I^q$ and $A = (\{\omega\} \times I^q) \cup (I^r \times \dot{I}^q)$, where $\dot{I}^q$ denotes the boundary of I^q. Again it is easy to see that X and A are contractible. We define $f : X \to B$ by

$$f(x_1,\ldots,x_r,y_1,\ldots,y_q) = u(x_1,\ldots,x_r) ,$$

and $g : A \to P$ by

$$g(x_1,\ldots,x_r,y_1,\ldots,y_q) = \begin{cases} v(y_1,\ldots,y_q), & \text{if } (x_1,\ldots,x_r,y_1,\ldots,y_q) \in \{\omega\} \times I^q \\ K(u,\lambda_i^\varepsilon v)(x_1,\ldots,x_r,y_1,\ldots,y_{q-1}) , & \varepsilon = 0,1, \\ & \text{if } (x_1,\ldots,x_r,y_1,\ldots,y_q) \in I^r \times \dot{I}^q. \end{cases}$$

We check that g is well defined

(1) Consider the point $a = (0,\ldots,0,y_1,\ldots,y_{i-1}, \varepsilon,y_i,\ldots,y_{q-1})$. $g(a) = v(y_1,\ldots,y_{i-1},\varepsilon,y_i,\ldots,y_{q-1}) = \lambda_i^\varepsilon v(y_1,\ldots,y_{q-1})$ and $g(a) = K(u,\lambda_i^\varepsilon v)(0,\ldots,0,y_1,\ldots,y_{q-1}) = FK(u,\lambda_i^\varepsilon v)(y_1,\ldots,y_{q-1})$. The two definitions of g at the point a are identical, since $\lambda_i^\varepsilon v = FK(u,\lambda_i^\varepsilon v)$ by the induction hypothesis.

(2) Consider the point $a' = (x_1,\ldots,x_r,y_1,\ldots,\varepsilon,\ldots,\varepsilon',\ldots,y_{q-2})$, where ε is in the i^{th} position and ε' is in the i'^{th} position, $i < i'$. The two definitions of g at this point are

$$g(a') = K(u,\lambda_{i-r}^\varepsilon v)(x_1,\ldots,x_r,y_1,\ldots, \varepsilon',\ldots,y_{q-2}) =$$
$$= \lambda_{i'-1}^{\varepsilon'} K(u,\lambda_{i-r}^\varepsilon v)(x_1,\ldots,y_{q-2}) \quad \text{and}$$
$$g(a') = K(u,\lambda_{i'-r}^{\varepsilon'} v)(x_1,\ldots,x_r,y_1,\ldots, \varepsilon,\ldots,y_{q-2})$$
$$= \lambda_i^\varepsilon K(u,\lambda_{i'-r}^{\varepsilon'} v)(x_1,\ldots,y_{q-2}).$$

Now the induction hypothesis gives

$\lambda_{i'-1}^{\varepsilon'} K(u,\lambda_{i-r}^\varepsilon v) = K(u,\lambda_{i'-r-1}^{\varepsilon'}\lambda_{i-r}^\varepsilon v)$ and $\lambda_i^\varepsilon K(u,\lambda_{i'-r}^{\varepsilon'} v) = K(u,\lambda_{i-r}^\varepsilon\lambda_{i'-r}^{\varepsilon'} v)$; hence the definitions of g are equivalent from (1.2).

Applying (3.3) we have the existence of a map $w : X \to P$ extending g and such that $p \circ w = f$. Since w extends g, $w = K(u,v)$ clearly satisfies conditions (1) and (2).

Condition 3: Should u or v be degenerate, we have defined g so that it will also be degenerate, hence we may collapse $A = (\{\omega\} \times I^q) \cup (I^r \times \dot{I}^q)$ along those coordinates on which g does not depend. We extend g to X as before. Clearly $w = K(u,v)$ does not depend on these coordinates, for otherwise it would contradict the fact that $w|A = g$. Hence $K(u,v)$ is degenerate and Lemma (3.18) is proven.

Returning to the proof of (3.17), we give the following definition.

(3.19) Definition. We define $\Psi : J_r \to E_r^o$ by the following formula:

$$\Psi(u \otimes v) = K(u,v),$$

where $K(u,v)$ is taken as an element of E_r^o.

In view of (3.18), condition (3), our definition of Ψ is clearly compatible with our definition of J_r. Since (3.18), condition (2), gives $\lambda_{i+r}^{\varepsilon}K(u,v) = K(u,\lambda_i^{\varepsilon}v)$, we can say that $d^r(K(u,v)) = K((-1)^r u,dv)$. In other words, $d^r \circ \Psi(u \otimes v) = \Psi \circ d_F(u \otimes v)$, which means that Ψ commutes with the boundary operators.

(3.20) Lemma. We now have $\phi \circ \Psi = 1$.

Proof: We need to show that $\phi \circ \Psi(u \otimes v) = u \otimes v$.

Now $\phi \circ \Psi(u \otimes v) = \phi(K(u,v))$, by (3.19)

$= B(K(u,v)) \otimes F(K(u,v))$, by (3.16)

$= u \otimes v$, by (3.18), condition 1.

It now remains to show that $h = \Psi \circ \phi$ is a homotopy operator on E_r^o.

(3.21) Lemma. To each cube u of P, of filtration $\leq r$ and dimension $n = r + q$, we can associate a cube Su of P, of filtration $\leq r$ and of dimension $n + 1$, such that

(1) $B \circ Su = Bu$

(2) $Su(0,\ldots,0,t,x_1,\ldots,x_q) = u(0,\ldots,0,x_1,\ldots,x_q)$

(3) $\lambda_{r+1}^{o}Su = u$ and $\lambda_{r+1}^{1}Su = K(Bu,Fu)$

(4) for all $i > r$; $\varepsilon = 0,1$; $S\lambda_i^{\varepsilon}u = \lambda_{i+1}^{\varepsilon}Su$

(5) if $q > 0$ and u is degenerate, then Su is degenerate.

Since the proof is similar to that of Lemma (3.18); namely by induction on the integer q and the use of (3.3), we do not give this proof. However, a proof may be found in [22].

(3.22) Definition. For $u \in E_r^o$ we define an operator k by the formula $k(u) = (-1)^r Su$.

k is well defined with respect to E_r^o, since (3.21) implies that, if $w(u) \leq r - 1$, then $w(Su) \leq r - 1$, and, if $u \in D^{(\infty)}$, then $Su \in D^{(\infty)}$.

Calculating $d^o ku + kd^o u$ we have

$$d^o ku = \sum_{i=r+1}^{n+1} (-1)^{i+r}(\lambda_i^1 Su - \lambda_i^o Su),$$

$$kd^o u = \sum_{i=r+1}^{n+1} (-1)^{i+r}(S\lambda_i^1 u - S\lambda_i^o u)$$

$$= \sum_{i=r+2}^{n+1} (-1)^{i+r+1}(\lambda_i^1 Su - \lambda_i^o Su) \qquad \text{from (3.21), condition 4,}$$

and so

$$d^o ku + kd^o u = \sum_{i=r+1}^{n+1} (-1)^{i+r}(\lambda_i^1 Su - \lambda_i^o Su) + \sum_{i=r+2}^{n+1} (-1)^{i+r+1}(\lambda_i^1 Su - \lambda_i^o Su)$$

$$= (-1)^{2r+1}(\lambda_{r+1}^1 Su - \lambda_{r+1}^o Su) \qquad \text{(other terms cancel).}$$

$$= -K(Bu, Fu) + u \qquad \text{from (3.21), condition 3,}$$

$$= u - \Psi \circ \phi(u) \qquad \text{from (3.16) and (3.19),}$$

$$= u - h(u)\ ;$$

hence h is a homotopy operator and proof is complete.

Now let G be an abelian group and filter the group $A \otimes G$, the group of chains in P with coefficients in G, by means of $A^r \otimes G$. The new E_r^o term is obtained by taking the tensor product of G and the E_r^o associated with the filtration of A. Now (3.17) shows that the new E_r^o is homotopically equivalent to $C_r(B) \otimes C(F) \otimes G = C_r(B) \otimes C(F;G)$.

Well known results, see for example [23], give that the homology groups of $C_r(B) \otimes C(F;G)$ are isomorphic to the groups $C_r(B) \otimes H_q(F;G)$, since $C_r(B)$ is a free group.

Hence we have proven

(3.23) <u>Theorem</u>. There is an isomorphism between the term $E^1_{r,q}$ of the spectral sequence attached to the filtration of $C(P;G)$, and the groups $C_r(B) \otimes H_q(F;G)$ induced by the homomorphism ϕ defined by (3.16). We also just write $E^1 \approx C(B,H(F))$.

<u>Term E^2</u>: We are now interested in finding out what the differential d^1 has been transformed into by the isomorphism in (3.23), and hence, obtain $E^2 = H(E^1)$. Since we will now need to use the homomorphisms ϕ, Ψ, B, F, and K for more than one value of r, we shall now index them with an upper right index to avoid any confusion which might arise.

We take $x = b \otimes h \in C_r(B) \otimes H_q(F;G)$ and consider a cycle c of the homology class of h; where c is of the form $c = \sum_\alpha g_\alpha u_\alpha$, $g_\alpha \in G$, $u_\alpha \in C(F)$. We now take an element $y \in C_r(B) \otimes C_q(F;G)$ which is a cycle of the homology class of x; clearly y may be of the form

$$y = b \otimes c = \textstyle\sum_\alpha g_\alpha b \otimes u_\alpha .$$

We apply Ψ to y and choose an element $z \in A^r$ which gives $\Psi(y)$ in passing to the quotient by A^{r-1}; in view of (3.19), z may be of the form $z = \sum_\alpha g_\alpha K^r(b,u_\alpha)$. Applying d, we have

(3.24) $$dz = \textstyle\sum_\alpha \sum_{i=1}^n (-1)^i g_\alpha(\lambda_i^1 K^r(b,u_\alpha) - \lambda_i^0 K^r(b,u_\alpha)),$$

which may be decomposed into two parts

(3.25) $\sum_\alpha\sum_{i\leq r}(-1)^i\, g_\alpha(\lambda_i^1 K^r(b,u_\alpha) - \lambda_i^o K^r(b,u_\alpha)) + \sum_\alpha\sum_{i>r}(-1)^i g_\alpha(\lambda_i^1 K^r(b,u_\alpha) - \lambda_i^o K^r(b,u_\alpha))$.

However, if $i > r$, (3.18), condition 2, gives $\lambda_i^\varepsilon K^r(b,u_\alpha) = K^r(b,\lambda_{i-r}^\varepsilon u_\alpha)$, $\varepsilon = 0,1$. Since we have assumed c to be a cycle, the expression $\sum_\alpha\sum_{i=1}^q g_\alpha(-1)^i(\lambda_i^1 u_\alpha - \lambda_i^o u_\alpha)$ must be a linear combination of degenerate cubes of F. The same holds true for the second partial sum of (3.25) by (3.18), condition 3. Hence this partial sum is null in $C(P)$ and we write

(3.26) $dz = \sum_\alpha\sum_{i=1}^r(-1)^i\, g_\alpha[\lambda_i^1 K^r(b,u_\alpha) - \lambda_i^o K^r(b,u_\alpha)]$.

Now each term of (3.26) is of filtration $\leq r - 1$, by (3.18) and the fact that each term is of the form $\lambda_i^\varepsilon K^r(b,u_\alpha), \varepsilon = 0,1$. We may obtain $dd = 0$ by direct computation and using (1.2); hence dz is a cycle.

By applying the operator ϕ^{r-1} to each term of (3.26), we obtain $\phi^{r-1}(dz)$, and subsequently obtain d^1X. To accomplish this in the light of (3.16), we need only consider the cubes $B^{r-1}\lambda_i^\varepsilon K^r(b,u_\alpha)$ and $F^{r-1}\lambda_i^\varepsilon K^r(b,u_\alpha)$, $i \leq r$, $\varepsilon = 0,1$. By (3.18), condition 1,

(3.27) $B^{r-1}\lambda_i^\varepsilon K^r(b,u_\alpha) = \lambda_i^\varepsilon b$,

and by (3.13)

(3.28) $F^{r-1}\lambda_i^\varepsilon K^r(b,u_\alpha)(x_1,\ldots,x_q) = K^r(b,u_\alpha)(0,\ldots,0,\varepsilon,0,\ldots,0,x_1,\ldots,x_q)$,

where ε is in the i^{th} place.

To interpret (3.28), we introduce, for all b, $i \leq r$, $\varepsilon = 0,1$, the construction $u \rightsquigarrow C(u)$ defined by

(3.29) $C(u)(t,x_1,\ldots,x_q) = K^r(b,u)(0,\ldots,0,t\varepsilon,0,\ldots,0,x_1,\ldots,x_q)$,

where $t\varepsilon$ is in the i^{th} place. Using (3.18) we see that $C(u)$ clearly satisfies (3.5) and is, therefore, a well defined construction for the loop $v(t) = b(0,\ldots,0,t\varepsilon,0,\ldots,0)$, where $t\varepsilon$ is in the i^{th} position.

We now denote by $S_{c,b,i,\varepsilon}$ the endormorphism of $C(P)$ associated to this construction, and we have $S_{c,b,i,\varepsilon}u_\alpha = F^{r-1}\lambda_i^\varepsilon K^r(b,u_\alpha)$ by (3.5) and (3.28). Now we obtain a cycle $t \in C_{r-1}(B) \otimes C_q(F;G)$, whose class will be equal to that of d^1x in $C_{r-1}(B) \otimes H_q(F;G)$; t may be written in the form

(3.30) $$t = \sum_\alpha \sum_{i=1}^{r} (-1)^i g_\alpha [(\lambda_i^1 b) \otimes S_{c,b,i,1}u_\alpha - (\lambda_i^0 b) \otimes S_{c,b,i,0}u_\alpha]$$

Denoting by $T_{b,i,\varepsilon}$ the automorphism of $H_q(F;G)$ defined by $S_{c,b,i,\varepsilon}$, we have from (3.6) that $T_{b,i,\varepsilon}$ depends only on the homotopy class of our loop $v(t)$. Now (3.30) yields

(3.31) $$d^1x = \sum_{i=1}^{r} (-1)^i (\lambda_i^1 b \otimes T_{b,i,1}h - \lambda_i^0 b \otimes T_{b,i,0}h).$$

Should the local system formed by $H_q(F;G)$ on B be trivial (3.31) becomes

(3.32) $$d^1x = db \otimes h ,$$

and we have proven

(3.33) Proposition: The isomorphism induced by ϕ of $E^1_{r,q}$ on $C_r(B) \otimes H_q(F;G)$ transform the differential d^1 into the natural boundary operator on $C_r(B)$.

We can now determine the values of the group $E^2_{r,q}$ since $E^2_{r,q} = H(E^1_{r,q})$; in fact, we have proven

(3.34) Theorem. There is an isomorphism between the terms $E^2_{r,q}$ of the Serre spectral sequence attached to the filtered complex $C(P;G)$ and the groups $H_r(B;H_q(F;G))$. We just write $E^2 \approx H(B;H(F))$.

(3.34) allows us to start with partial information on the homology groups of P, B, and F, or in other words, partial information on E^2 and E^∞, and argue towards more complete knowledge. The passage from $E^2_{r,q}$ to $E^\infty_{r,q}$ is effected by repeatedly taking quotient groups of subgroups.

<u>The transgression</u>. We will now introduce a property of the spectral sequence of a fiber space : the concept of transgression.
This concept may be introduced in the general situation for spectral sequences, but we only need it in a particular case. The concept of transgression has been dealt with by [22], [12], [23], and elsewhere.

Although there are several known definitions, we shall only give two classical ones, see [15] or [22].

(3.35) <u>Definition</u>. The <u>transgression</u> T is defined to be the differential $T = d^n : E^n_{n,0} \to E^n_{0,n-1}$.

For $n \geq 2$, T maps a certain subgroup of $H_n(B;G)$ into a certain quotient of $H_{n-1}(F;G)$.

This statement may be understood in the following manner:

For $n \geq 2$, there can be no non zero element of $E^p_{n,0}$ asa boundary for d^p, since d^p increases the complimentary degree; hence we have a sequence of monomorphisms

$$E^n_{n,0} \to \cdots \to E^3_{n,0} \to E^2_{n,0} = H_n(E^1_{n,0}) \quad ,$$

and $\quad H_n(E^1_{n,0}) = H_n(B;G)$ from (3.23).

For $n \geq 1$, all the elements of $E^p_{0,n-1}$ are cycles for d^p, since d^p reduces the filtered degree; hence we have a sequence of epimorphisms

$$E^1_{0,n-1} \to E^2_{0,n-1} \to \cdots \to E^n_{0,n-1} \quad ,$$

and $\quad E^1_{0,n-1} = H_n(A^o)$ from (2.15). Now A^o is formed by elements of $A = C(P)$ of null fibration with vertices at x, so that it is contained in the fiber F passing by x. Thus, $A^{o,n} \approx C_n(F;G)$.

(3.36) Definition. Given, for $n \geq 2$, the two homomorphisms

$$H_{n-1}(F;G) \xleftarrow{d} H_n(P/F;G) \xrightarrow{p_*} H_n(B;G),$$

we denote by M the kernel of p_* and by M' the image of p_*. Now for $x \in M'$ and $y \in H_n(P/F;G)$ such that $p_*(y) = x$, we have $d(y) \in H_{n-1}(F;G)$ such that, as y varies, $d(y)$ describes a class modulo $d(M)$. By passing to quotient we have a homomorphism

$$T : M' \to H_{n-1}(F;G)/d(M).$$

called the transgression.

(3.37). The elements of M' may be called the transgressive elements of $H_n(B;G)$; and a cycle of $(B;G)$, of which a homology class is transgressive, is called a transgressive cycle. Translating into terms of chains, we see that, for a cycle x of B to be transgressive, it is necessary and sufficient that there exist a chain y on P, projected on x by $P \to B$, and such that dy is a chain of F.

We now state a result, the proof of which was stated in [15] and detailed in [22].

(3.38) Proposition. The groups M' and $H_{n-1}(F;G)/d(M)$ are isomorphic to the groups $E^n_{n,0}$ and $E^n_{0,n-1}$, respectively; by this isomorphism (3.35) and (3.36) are equivalent.

Among other things this tells us that $E^n_{n,0}$ is the image of $H_n(P/F;G)$ in $H_n(B;G)$ by the projection.

We now consider a diagram due to Serre [22], page 452.

(3.39)
$$\begin{tikzcd}
 & E^n_{n,0} \arrow[dl] \arrow[rrr, "d_n"] & & & E^n_{0,n-1} \\
H_n(B;G) & & H_n(P/F;G) \arrow[ll, "p_*"'] \arrow[ul] \arrow[rr, "d"] & & H_{n-1}(F;G) \arrow[u] \\
\Pi_n(B) \otimes G \arrow[u] & & \Pi_n(P/F) \otimes G \arrow[u] \arrow[ll] \arrow[rr] & & \Pi_{n-1}(F) \otimes G \arrow[u]
\end{tikzcd}$$

(3.40) Lemma. (3.39) is a commutative diagram.

Clearly the top half commutes because of (3.38).

Now the homomorphism from $\Pi_n(X) \otimes G \to H_n(X;G)$, as used above, is just the composite $i \circ (h \times 1)$, where h is a Hurewicz homomorphism and i is the natural embedding of $H_n(X) \otimes G$ into $H_n(X;G)$ as a direct summand with factor group $\mathrm{Tor}_{n-1}(X,G)$ (from the Universal Coefficient Theorem, see [13]). Now commutativity of the bottom half of (3.39) follows from the fact that h commutes with boundaries and induced maps, see [23], Theorem 3, page 389.

(3.41) Definition. A cycle whose homology class is in the image of the Hurewicz homomorphism is called a spherical cycle, and its class a spherical homology class.

(3.42) Lemma. Each spherical homology class of B is transgressive.

Proof. Corollary 9, page 377 of [23], gives us that the homomorphism $\Pi_n(P/F;G) \to \Pi_n(B;G)$ is an isomorphism. Then from (3.39) we see that the image of $\Pi_n(B) \otimes G$ in $H_n(B;G)$ is contained in $E^n_{n,0}$. Now using the definition of transgression, we have the required result.

We now give the proof of (3.6).

Proof. We let v be a loop on B and put $C(u) = K(u,v)$; clearly the conditions 1 - 3 of (3.18) imply that the required conditions 1 - 4 are satisfied. Hence there is at least one such construction.

To complete the proof of this lemma, we introduce another lemma.

(3.43) Lemma. Given $h : I^2 \to B$ is a 2 dimensional cube of B, such that $h(0,t') = h(1,t') = b$, for all $t' \in I$; if we put $v_1(t) = h(t,0)$ and $v_2(t) = h(t,1)$, v_1 and v_2 are homotopic loops of B. We take C_1 and C_2 to be constructions for the loops v_1 and v_2 , respectively. Now for each n-dimensional cube u of F, there exist a cube of F, say Hu , of dimension $n + 2$ and of filtration ≤ 2, such that

(1) $BHu = h$

(2) $\lambda_1^o Hu(t,y_1,\ldots,y_n) = u(y_1,\ldots,y_n)$

(3) $\lambda_2^o Hu = C_1 u$, $\lambda_2^1 Hu = C_2 u$

(4) $H\lambda_i^\varepsilon u = \lambda_{i+2}^\varepsilon Hu$, $\varepsilon = 0,1$, $1 \leq i \leq n$.

(5) if $u \in D^{(\infty)}$ then $Hu \in D^{(\infty)}$.

Proof of (3.43).

The proof is by induction on the integer n.

Case 1 : n = 0.

Since there is only one 0-cube, the cube u is reduced to the point x. We transform our problem into that of covering a map.

Here our cube Hu will be a 2-dimensional cube of F. We put $X = I^2$ and $A = (I \times \{0\}) \cup (I \times \{1\}) \cup (\{0\} \times I)$; clearly both X and A are contractible.

We define $f : X \to B$ by putting $f = h$, and $g : A \to P$ by

$$\begin{cases} g(t,0) = C_1u(t) \quad , & (t,0) \in I \times \{0\} \\ g(t,1) = C_2u(t) \quad , & (t,1) \in I \times \{1\} \\ g(0,t) = x \quad , & (0,t) \in \{0\} \times I. \end{cases}$$

Clearly g is well defined and we are able to apply (3.3), thereby obtaining a map $w : X \to P$ such that $p \circ w = f$ and which extends g. We put $Hu = w$. Since w extends g, the definition of g insures that Hu satisfies the required conditions.

Case 2 : from n - 1 to n.

We assume that $n \geq 1$ and that, for all $n' < n$, we have already constructed Hu satisfying the required conditions. We now construct Hu when u is of dimension n. The method we shall use is to transform this problem into that of covering a map.

We put $X = I^2 \times I^n$ and $A = (I \times \{0\} \times I^n) \cup (I \times \{1\} \times I^n) \cup (\{0\} \times I \times I^n) \cup (I \times I \times \dot{I}^n)$. Clearly A and X are contractible.

We define $f : X \to B$ by

$$f(t,t',x_1,\ldots,x_n) = h(t,t'),$$

and $g : A \to P$ by

$$\begin{cases} g(t,0,x_1,\ldots,x_n) = C_1u(t,x_1,\ldots,x_n), \text{ if } (t,0,x_1,\ldots,x_n) \in I \times \{0\} \times I^n \\ g(t,1,x_1,\ldots,x_n) = C_2u(t,x_1,\ldots,x_n), \text{ if } (t,1,x_1,\ldots,x_n) \in I \times \{1\} \times I^n \\ g(0,t,x_1,\ldots,x_n) = u(x_1,\ldots,x_n) \quad , \text{ if } (0,t,x_1,\ldots,x_n) \in \{0\} \times I \times I^n \\ g(t,t',x_1,\ldots,x_{i-1},\varepsilon,x_i,\ldots,x_{n-1}) = H\lambda_i^\varepsilon u(t,t',x_1,\ldots,x_{n-1}), \text{ if} \\ \qquad (t,t',x_1,\ldots,x_{i-1},\varepsilon,x_i,\ldots,x_{n-1}) \in I \times I \times \dot{I}^n \end{cases}$$

It is not difficult to show that g is well defined, so we omit the details and just remark that it follows from the definitions, the induction hypothesis, and, in one case, from (1.2).

We apply (3.3) and obtain a map $w : X \to P$ which extends g and is such that $p \circ w = f$. We put $Hu = w$.

Since Hu extends g, the first four conditions follow immediately. If $u \in D^{(\infty)}$, we have so defined our g that it is also degenerate. Before extending g from A to X we collapse A along the coordinates on which g does not depend. Hence $Hu = w$ will also have to be degenerate or else the condition $w|A = g$ would not hold.

Hence (3.43) is proven.

We return to the proof of (3.6).

We consider $S_{C_1} = S_1$, $S_{C_2} = S_2$, obtained from C_1, C_2, in the manner of (3.5), and define an $n + 1$ dimensional endomorphism $k(u)$ of chains of F by

$$k(u)(t,x_1,\dots,x_n) = Hu(1,t,x_1,\dots,x_n).$$

Now calculating $dku + kdu$ we have

$$dku = \sum_{i=1}^{n}(-1)^{i}(\lambda_i^1 ku - \lambda_i^0 ku)$$

$$= \sum_{i=1}^{n}(-1)^{i}(\lambda_{i+1}^1 Hu - \lambda_{i+1}^0 Hu) ,$$

$$kdu = \sum_{i=1}^{n}(-1)^{i}(H\lambda_i^1 u - H\lambda_i^0 u)$$

$$= \sum_{i=1}^{n}(-1)^{i}(\lambda_{i+2}^1 Hu - \lambda_{i+2}^0 Hu) ,$$

by (3.43), condition 4,

and
$$dku + kdu = \sum_{i=1}^{n}(-1)^{i}(\lambda_{i+1}^1 Hu - \lambda_{i+1}^0 Hu) + \sum_{i=1}^{n}(-1)^{i}(\lambda_{i+2}^1 Hu - \lambda_{i+2}^0 Hu)$$

$$= \lambda_2^0 Hu - \lambda_2^1 Hu,$$ since all other terms cancel.

Now (3.43), condition 3, implies that

$$dku + kdu = C_1u - C_2u.$$

Since $S_{C_i}u(x_1,\ldots,x_n) = C_iu(1,x_1,\ldots,x_n)$ by (3.5), we have

$$dku + kdu = S_1u - S_2u\ .$$

Hence S_1 and S_2 are homotopically equivalent and we have proven (3.6).

CHAPTER 3 APPENDIX

Applications

We will now examine some general applications of the spectral sequence of a fibration.

We recall that, for a field G, $H_i(X;G)$ is a vector space over G and that we may define the Euler characteristic χ of the space X in the following manner

$$\chi(X) = \sum_i (-1)^i x_i \quad ,$$

where x_i, the Betti number (dimension of $H_i(X;G)$), is finite for all i and zero for i large enough.

Relating this to a fibration, as defined by (3.2), we have

(3a.1) Proposition. When G is a field and

(1) the local system formed by $H_i(F;G)$ on B is trivial for all $i \geq 0$, and

(2) the Betti numbers b_i and f_i (of B and F, respectively) are finite for all i and zero for i large enough, we have the Euler characteristics of P, B and F satisfying the relation

$$\chi(P) = \chi(B) \cdot \chi(F) \ .$$

Proof. By the Universal Coefficient Theorem (see [13] for a definition) and the fact that G is a field,

$$E^2 = H(B;G) \otimes H(F;G) \quad ,$$

and hence is of finite dimension, since both $H(B;G)$ and $H(F;G)$ are of finite dimension. Now by (2.13)' , $E^2, \ldots, E^\infty$ are finite dimensional graded vector spaces over G; hence, we may consider the Euler characteristic of these terms.

Condition (2) in the statement of (3a.1) implies that d^P will be trivial for p large enough; therefore, $E^P \approx E^\infty$ and

$$\chi(P) = \chi(E^\infty) = \chi(E^P)$$

for sufficiently large p. From the well known properties of tensor products, we have

$$\chi(E^2) = \chi(H(B;G) \otimes H(F;G)) = \chi(B) \cdot \chi(F).$$

Using (2.13)'

$$\chi(E^{P+1}) = \chi(E^P) .$$

Combining the above results, we have

$$\chi(B) \cdot \chi(F) = \chi(E^2) = \ldots = \chi(E^P) = \chi(E^\infty) = \chi(P).$$

Should B be a finite polyhedron, we could relax the conditions of (3a.1); in fact, we may drop condition (1) completely, see [22] for the details.

This application of the spectral sequence of a fiber space shows one of the relationships which exist between the three spaces P, B, and F. Another general relationship has been given by [17] and proven in detail by [22]. We merely state the result.

(3a.2) <u>Proposition</u>. If G is a principal ideal domain, and the local system formed by $H_i(F;G)$ on B is trivial for all i, then, if the homology groups (with coefficients in G) of two of the three spaces P, B, and F are finitely generated, so is the third.

If we place certain restrictions on one or more of the three spaces P, B, and F, we will obtain stronger results. Consider, for example, the situation in which either F or B is a homology n-sphere.

(3a.3) Proposition. If G is a principal ideal domain and if B is a simply connected r-sphere, $r > 1$, we have the exact sequence

$$\ldots \to H_{i+1}(P;G) \to H_{i-r+1}(F;G) \to H_{i-r}(F;G) \to H_i(P;G) \to \ldots$$

Proof. We give a proof which is similar to that given in [11], page 432, for another result.

(3a.4) Since B is a homology r-sphere, we have, from (3.34), that $E^2_{p,q} = 0$ for $p \neq 0,r$ and, consequently, that $E^k_{p,q} = 0$ for $p \neq 0,r$.

From (2.18)'

(3a.5) $$0 \subseteq D_{0,i} \subseteq \cdots \subseteq D_{r-1,i-r+1} \subseteq D_{r,i-r} \subseteq \cdots \subseteq D_{i,0} = H_i(P;G).$$

Now (3a.4) implies that, for $i > r$, all quotients except $D_{r,i-r}/D_{r-1,i-r+1}$ and $D_{0,i}/0$ are zero.

Hence we have

$$E^\infty_{0,i} \approx D_{0,i} \approx \ldots \approx D_{r-1,i-r+1}$$

and

$$E^\infty_{r,i-r} \approx D_{r,i-r}/D_{r-1,i-r+1} \approx \ldots \approx H_i(P;G)/D_{r-1,i-r+1} \quad ;$$

this gives us the exact sequence

(3a.6) $$0 \to E^\infty_{0,i} \to H_i(P;G) \to E^\infty_{r,i-r} \to 0.$$

Now (3a.4) implies that the only non-zero differential is of the form

$$d^r : E^r_{r,i} \to E^r_{0,i+r-1} \quad .$$

For $k \geq 2$, every element of $E^r_{0,i+r-1}$ is a cycle for d^k by (2.25), since (3.9) implies that the filtration of C(P) is regular. For $2 \leq k < r$, our hypothesis implies $E^k_{k,i+r-k} = 0$, so that no non-zero element of $E^k_{0,i+r-1}$ is a boundary for d^k. Hence

$$E^2_{0,i+r-1} \approx \dots \approx E^r_{0,i+r-1} \quad .$$

(2.27) implies that

$$E^{r+1}_{0,i+r-1} \approx \dots \approx E^{\infty}_{0,i+r-1} \quad ,$$

since the filtration of $C(P)$ is regular. Therefore, there is an exact sequence

$$\begin{array}{ccccccc} E^r_{r,i} & \xrightarrow{d^r} & E^r_{0,i+r-1} & \longrightarrow & E^{r+1}_{0,i+r-1} & \longrightarrow & 0 \\ & & \wr\wr & & \wr\wr & & \\ & & E^2_{0,i+r-1} & & E^{\infty}_{0,i+r-1} & & \end{array} \tag{3a.7}$$

Using similar methods, we have $E^{r+1}_{r,i} \approx E^{\infty}_{r,i}$, $E^{r+1}_{r,i}$ mapped momomorphically into $E^r_{r,i}$, and $E^r_{r,i} \approx E^2_{r,i}$; consequently there is an exact sequence

$$\begin{array}{ccccccc} 0 & \to & E^{r+1}_{r,i} & \to & E^r_{r,i} & \xrightarrow{d^r} & E^r_{0,i+r-1} \\ & & \wr\wr & & \wr\wr & & \\ & & E^{\infty}_{r,i} & & E^2_{r,i} & & \end{array} \tag{3a.8}$$

Now, combining (3a.7) and (3a.8), we have an exact sequence

$$0 \to E^{\infty}_{r,i} \to E^2_{r,i} \xrightarrow{d^r} E^2_{0,i+r-1} \to E^{\infty}_{0,i+r-1} \to 0 \ . \tag{3a.9}$$

Using (3a.9) and (3a.6) we have an exact sequence

$$\dots \to H_{i+1}(P;G) \to E^2_{r,i-r+1} \to E^2_{0,i} \to H_i(P;G) \to \dots \quad . \tag{3a.10}$$

Since B is a homology r-sphere, (3.34) implies that, for total degree i given, the non-zero terms of E^2 are

(3a.11)
$$\begin{cases} E^2_{0,i} \approx H_o(B;\ H_i(F;G)) \approx H_i(F;G) \\ E^2_{r,i-r} \approx H_r(B;H_{i-r}(F;G)) \approx H_{i-r}(F;G) \end{cases}$$

Substituting (3a.11) in (3a.10) we obtain the required exact sequence.

This sequence is called the Wang Exact Sequence. There are several proofs available, see, for example [27], [22], or [23].

Placing the restriction of (3a.3) on the space F instead of on the space B, we obtain a similar result.

(3a.12) Proposition. If G is a principal ideal domain and F is a homology s-sphere, $s \geq 1$, such that the local system formed by $H_s(F;G)$ on B is trivial, there is an exact Gysin sequence

$$\cdots \to H_{i+1}(B;G) \to H_{i-s}(B;G) \to H_i(P;G) \xrightarrow{P_*} H_i(B;G) \to \cdots$$

We do not give a proof for (3a.12) as the proof would directly parallel that of (3a.3); however, proofs of (3a.12) may be found in [8], [22], and others.

Another interesting situation arises when both F and B are homology spheres; for example, F could be a homology s-sphere, $s \geq 1$, and B could be a homology r-sphere, $r \geq 2$. In such a case we may use either the Wang or the Gysin Exact sequence to directly calculate the homology groups $H_i(P)$. This has been done in [12] using the Wang Exact Sequence.

The method of proof used in (3a.3) and suggested for (3a.12) may be also used when there are certain other restrictions on the spaces P, B, and F. In fact, this approach was employed by Hilton and Wylie [11] to prove the following proposition:

(3a.13) <u>Proposition</u>. If G is a principal ideal domain, we assume that the local system formed by $H_i(F;G)$ on B is trivial for all $i \geq 0$, that $H_i(B;G) = 0$, for $0 < i < p$, and that $H_i(F;G) = 0$, for $0 < i < q$. Under these conditions we have the exact sequence

$$H_{p+q-1}(F;G) \to H_{p+q-1}(P;G) \to H_{p+q-1}(B;G)$$
$$\overset{T}{\to} H_{p+q-2}(F;G) \to \ldots \to H_2(B;G) \overset{T}{\to} H_1(F;G)$$
$$\to H_1(P;G) \to H_1(B;G) \overset{T}{\to} 0 \quad ,$$

where $T : H_n(B;G) \to H_{n-1}(F;G) \quad , \quad 0 < n \leq p + q - 1,$

is the transgression.

Placing more restrictions on our spaces, we have the following corollaries:

(3a.14) <u>Corollary</u>. If $H_i(B;G) = 0$ for all $i > 0$, the homomorphism $H_i(F;G) \to H_i(P;G)$ is an isomorphism.

Proof follows from (3a.13) by putting $p = \infty$, $q = 1$.

(3a.15) <u>Corollary</u>. If $H_i(F;G) = 0$ for all $i > 0$, the projection $p : P \to B$ defines an isomorphism of $H_i(P;G)$ onto $H_i(B;G)$ for all $i \geq 0$.

Proof is immediate if we put $p = 1$, $q = \infty$ in (3a.13).

There are a number of other useful applications of the spectral sequence of a fiber space. In the next chapter we shall use this spectral sequence to obtain results concerning the Pontryaginproduct in a loop space.

CHAPTER 4

The Pontryagin Product in a Loop Space.

In 1939, Pontryagin [18] introduced a multiplication in the homology groups of certain topological groups; the homology product being induced from the group operation in the topological group. Today the term "Pontryagin Product" has come to refer to the homology product which may be induced from the multiplication operator of an H-space.

(4.1) Definition. An H-space consists of a pointed space X with a continuous multiplication operator $m : X \times X \to X$ for which the constant map $c : X \to X$ is a homotopy identity; that is, the compositions

$$X \xrightarrow{(c,1)} X \times X \xrightarrow{m} X \quad \text{and} \quad X \xrightarrow{(1,c)} X \times X \xrightarrow{m} X$$

are homotopic to the identity 1_X.

An H-space is called an H-group if the multiplication m is homotopy associative and if there exists a homotopy inverse for m.

It is well known that both topological groups and loop spaces are H-groups and that the Pontryagin multiplication may be defined for both. Here we shall consider in detail the Pontryagin multiplication for a path space. We shall use the concepts of cubical homology in our approach, which is that of [3], although the concept of singular simplical homology may be equally well used (see [11], page 359) for the definitions of the product. However, later considerations lend themselves more conveniently to the use of the cubical concepts.

Recalling the results of Chapter 1, we consider a map

(4.2) μ from $Q(X) \otimes Q(Y)$ into $Q(X \times Y)$

induced by the association of a p-dimensional cube u in an arcwise connected space X and a q-dimensional cube v in an arcwise connected space Y with the $(p + q)$-dimensional cube $u \times v$ in the cartesian product $X \times Y$, defined by

$$(4.3) \qquad u \times v(x_1,\ldots,x_{p+q}) = (u(x_1,\ldots,x_p),\ v(x_{p+1},\ldots,x_{p+q})).$$

The tensor product $Q(X) \otimes Q(Y)$ will be equipped with the usual differential $d = d \otimes 1 + w \otimes d$, where $w(x) = (-1)^q x$, for $x \in Q_p(X)$. Clearly d has the property that $dd = 0$, since the differentials on $Q(X)$ and $Q(Y)$ have this property; while (4.3) allows us to write $x \times y$ for $\mu(x \otimes y)$. Now $\mu d(x \otimes y) = \mu(dx \otimes y + wx \otimes dy)$
$= \mu(dx \otimes y) + \mu(wx \otimes dy) = dx \times y + wx \times dy = d(x \times y) = d\mu(x \otimes y)$; so that

(4.4) $\quad d$ and μ commute.

(4.3) implies that, should u or v be degenerate, then $u \times v$ is also degenerate, so that μ is compatible with the definition (1.10). Hence we have a map, also called μ, from $C(X) \otimes C(Y)$ into $C(X \times Y)$, defined by

$$(4.5) \qquad \mu([x] \otimes [y]) = [\mu(x \otimes y)].$$

Now (4.4) and (4.5) imply that this μ also commutes with the differentials. In addition, Bott and Samelson [3] have shown that $\mu : C(X) \otimes C(Y) \to C(X \times Y)$ is a chain equivalence.

By passing to homology, μ induces a homomorphism $\mu_* : H(X) \otimes H(Y) \to H(X \times Y)$; should the coefficients be in a principal ideal domain, the Künneth formula (see [13], for a definition) can be applied and μ_* imbeds $H(X) \otimes H(Y)$ into $H(X \times Y)$ as a direct summand.

The results of Chapter 1 allow us to choose a point x_o in an arcwise and simply connected space X and to say that the vertices of all singular cubes are at this point.

(4.6) <u>Definition</u>. The space P of paths in X may be defined in the following manner;

$$P = \{f : I \to X \mid f(1) = x_o\} .$$

(4.7) <u>Definition</u>. The space Ω of loops in X may be defined in the following manner:

$$\Omega = \{f : I \to X \mid f(0) = f(1) = x_o\} .$$

Clearly both spaces may be equipped with the compact open topology (for a definition see [14]); hence, in future, we will assume both P and Ω to have this topology.

(4.8) <u>Definition</u>. We define a projection $p : P \to X$ by the formula $p(f) = f(0)$.

(4.9) <u>Definition</u>. We define a multiplication $m : P \times \Omega \to P$ in the following manner: For $x \in P$, $y \in \Omega$, the composition $x \circ y$ of paths x and y may be defined by

$$x \circ y(t) = \begin{cases} x(2t) , & 0 \leq t \leq \frac{1}{2} \\ y(2t - 1) , & \frac{1}{2} \leq t \leq 1 \end{cases} .$$

We put $m(x,y) = x \circ y$.

Since, for $t = \frac{1}{2}$, $x(2(\frac{1}{2})) = x(1) = x_o = y(0) = y(2(\frac{1}{2}) -1)$, we have that m is well defined. Because $x \circ y(0) = x(0)$, we have

(4.10) $\quad p \circ m(x,y) = p(x \bullet y) = p(x)$

(4.11) Definition. We define a constant loop $e : I \to X$ by

$$e(t) = x_o \qquad , \quad t \in I .$$

Now the results of Chapter 1 allow us to say that the vertices of all cubes in P and Ω may be at e.

Our multiplication $m : P \times \Omega \to \Omega$ clearly induces a map

(4.12) $\quad m^{\#} : Q(P \times \Omega) \to Q(P),$

which from (4.9), is obviously compatible with the definition (1.10) and commutes with the differentials.

(4.13) Definition. We define a chain map

$$\zeta : Q(P) \otimes Q(\Omega) \to Q(P)$$

to be the composite

$$Q(P) \otimes Q(\Omega) \xrightarrow{\mu} Q(P \times \Omega) \xrightarrow{m^{\#}} Q(P),$$

where μ and $m^{\#}$ are defined by (4.2) and (4.12), respectively.

(4.14) Lemma. ζ commutes with the differentials.

Proof. Proof is immediate from (4.4) and (4.12)

(4.14) implies that we also have a homomorphism $\zeta_* = m_* \circ \mu_*$ of the homology groups.

(4.15) Definition. We define $u * v = \zeta(u \otimes v)$ and $z * w = \zeta_*(z \otimes w)$ to be the Pontryagin Multiplication.

(4.16) Remark. (4.9) implies that, if we restrict m to $\Omega \times \Omega$, we have a map from $\Omega \times \Omega$ into Ω . We use no special notation for this special case.

(4.17) Lemma. The constant loop e has the property $e \cdot e = e$.

The proof is obvious from (4.11).

(4.18) Proposition. The compositions

$$i_r : P \to P \times \Omega \to P, \qquad i_\ell : \Omega \to \Omega \times \Omega \to \Omega ,$$

defined by $i_r(x) = m(x,e) = x \cdot e$,

and $i_\ell(x) = m(e,x) = e \cdot x$, are homotopic to the identity.

Proof. Case 1 : i_r .

We take $f \in P$, and define a family of paths f_θ ; $0 \le \theta \le 1$, by the formula

$$f_\theta(t) = \begin{cases} f(\frac{2t}{\theta + 1}) & , \quad 0 \le t \le \frac{\theta + 1}{2} \\ x_o & , \quad \frac{\theta + 1}{2} \le t \le 1 \end{cases} .$$

This formula gives us that $f_\theta(1) = x_o$, so that f_θ is a path. Since $f_o = i_r$ and $f_1 = 1_P$, the fact that $(f,\theta) \leadsto f_\theta$ is clearly a continuous mapping of $P \times I \to P$ implies that i_r is homotopic to 1_P with stationary e.

Case 2 : i_ℓ .

We take $f \in \Omega$, and define a family of loops f_θ ; $0 \le \theta \le 1$, by the formula

$$f_\theta(t) = \begin{cases} x_o & , \quad 0 \le t \le \theta/2 \\ f(\frac{2t - \theta}{2 - \theta}) & , \quad \theta/2 \le t \le 1 . \end{cases}$$

Now this fromula gives $f_\theta(0) = f_\theta(1) = x_o$, so that f_θ is a loop. Since $f_o = 1_\Omega$ and $f_1 = i_\ell$, the fact that $(f,\theta) \leadsto f_\theta$ is clearly a continuous mapping of $\Omega \times I \to \Omega$ implies that i_ℓ is homotopic to 1_Ω with stationary e.

We also note that (4.16) implies that i_r restricted to Ω is homotopic to the identity on Ω.

(4.19) Proposition. The multiplication m is homotopy associative.

Proof. We consider two maps g_o and $g_1 : P \times \Omega \times \Omega \to P$ defined by $g_o(q_1,q_2,q_3) = (q_1 \cdot q_2) \cdot q_3$ and $g_1(q_1,q_2,q_3) = q_1 \cdot (q_2 \cdot q_3)$ and define a homotopy f_θ, $0 \leq \theta \leq 1$, by the formula

$$f_\theta(q_1,q_2,q_3)(t) = \begin{cases} q_1\left(\dfrac{4t}{\theta+1}\right) & , \quad 0 \leq t \leq \dfrac{\theta+1}{4} \\ q_2(4t-\theta-1) & , \quad \dfrac{\theta+1}{4} \leq t \leq \dfrac{\theta+2}{4} \\ q_3\left(\dfrac{4t-\theta-2}{2-\theta}\right) & , \quad \dfrac{\theta+2}{4} \leq t \leq 1 \end{cases} .$$

Now this formula implies that f_θ has the following properties:

(1) $f_o = g_o$, $f_1 = g_1$.

(2) $p \circ f_\theta(q_1,q_2,q_3) = p(q_1)$, for all q_1,q_2,q_3,θ .

(3) $f_\theta(e,e,e) = e$, for all θ .

These properties together with the usual continuity relation imply that $g_o \simeq g_1$; hence m is a homotopy associative.

(4.16) implies that m is still homotopy associative in the restrictive case.

(4.20) Proposition. The space of loops has homotopy inverses.

Proof. We define a map $\gamma : \Omega \to \Omega$ by the formula $\gamma(f)(t) = f(1-t)$ and claim that $\gamma(f)$ is a homotopy inverse for f, $f \in \Omega$.

With $f \in \Omega$, we define a family of loops f_θ, $0 \leq \theta \leq 1$, by

$$f_\theta(t) = \begin{cases} x_o & , \quad 0 \leq t \leq \theta/2 \\ f(2t-\theta) & , \quad \theta/2 \leq t \leq \frac{1}{2} \\ f(2-2t-\theta) & , \quad \frac{1}{2} \leq t \leq (1-\theta/2) \\ x_o & , \quad (1-\theta/2) \leq t \leq 1 \end{cases}$$

This formula gives $f_\theta(0) = f_\theta(1) = x_o$; consequently f_θ is a loop. Also $f_1 = e$ and $f_o = f \cdot \gamma(f)$, so that the obvious continuity relation implies that the space of loops has homotopy inverses.

It now follows immediately from (4.17) - (4.20) that Ω is an H-group. The H-group structure induces a group structure in $H(\Omega)$, so we may speak of the Pontryagin algebra, where the zero class, defined and denoted by e, is the unit.

We state a well known proposition.

(4.21) Proposition. The triple (P,p,X) is a fiber space with fiber Ω .

A proof of this proposition may be found in [22] and others.

Now (4.21) and (3.9) tell us that $Q(P)$; and subsequently, by factoring out the degenerate cubes, $C(P)$ may be filtered, giving rise to a spectral sequence. We shall use the terms of the spectral sequence determined in Chapter 3.

(4.22) Lemma. For $u \in Q_p(P)$ and $v \in Q_q(\Omega)$, the filtration of the Pontryagin product $u * v$ is equal to the filtration of u.

Proof. From (4.10) $p(x \cdot y) = p(x)$, so that

$$p(u(x_1,\ldots,x_p) \cdot v(x_{p+1},\ldots,x_{p+q})) = p(u(x_1,\ldots,x_p)).$$

The result follows from (3.9).

(4.23) Definition. We put $\Lambda = C(P) \otimes C(\Omega)$ and define a differential d on Λ by

$$d = d_p \otimes 1 + w \otimes d_\Omega ,$$

where $w(x) = (-1)^p x$, for $x \in C(P)$.

Since $A^P(P)$ is a direct summand of $C(P)$, we may filter $C(P) \otimes C(\Omega)$ by the subgroups $A^P(P) \otimes C(\Omega)$; so that Λ may be filtered by the subgroups $A^P(\Lambda) = A^P(P) \otimes C(\Omega)$. The definition of ζ and (4.9) imply that if we restrict ζ to $A^P(P) \otimes C(\Omega)$, we have a map

(4.24) $$\zeta : A^P(\Lambda) = A^P(P) \otimes C(\Omega) \to A^P(P).$$

Now (4.24) induces maps

(4.25) $$\zeta^r : E^r(\Lambda) \to E^r(P) .$$

Together, the ζ^r form a map ζ of the spectral sequence of Λ into that of P. Since the ζ^r are induced from the original ζ, they obviously commute with the differentials d^r.

(4.26) Lemma. There is an isomorphism

$$\alpha^o : E^o(P) \otimes C(\Omega) \to E^o(\Lambda).$$

Proof. By (2.15) , $E^o(\Lambda) = \sum_P \frac{A^P(\Lambda)}{A^{P-1}(\Lambda)}$

$$= \sum_P \frac{A^P(P) \otimes C(\Omega)}{A^{P-1}(P) \otimes C(\Omega)}$$

and $E^o(P) \otimes C(\Omega) = \sum_P \frac{A^P(P)}{A^{P-1}(P)} \otimes C(\Omega)$.

Since $A^{P-1}(P)$ is a direct summand of $A^P(P)$, we have $\alpha^o : E^o(P) \otimes C(\Omega) \approx E^o(\Lambda)$ with respect to the differentials $d^o \otimes 1 + w \otimes d_\Omega$, where $w(x) = (-1)^P x$, $x \in E^o(P)$, and d^o.

Now using the notations of Chapter 2, and letting Z be the group of cycles and B be the group of boundaries of $C(\Omega)$, we see that the identity map of $C(P) \otimes C(\Omega) \to \Lambda$ induces maps

(4.27) $\alpha^r : Z_p^r(P) \otimes Z \to Z_p^r(\Lambda)$, $r \geq 1$.

(4.28) Proposition. The maps α^r induce a map

$$\alpha^r : E^r(P) \otimes H(\Omega) \to E^r(\Lambda) \quad , \quad r \geq 1,$$

such that α^r takes the differential $d^r \otimes 1$ to the differential d^r.

Proof. We want to show that α^r is compatible with the definition of E^r. To do this we must show three things:

(1) $\alpha^r(Z_p^r(P) \otimes B) \subset Z_{p-1}^{r-1}(\Lambda) + B_p^{r-1}(\Lambda)$,

(2) $\alpha^r(Z_{p-1}^{r-1}(P) \otimes Z) \subset Z_{p-1}^{r-1}(\Lambda)$,

(3) $\alpha^r(B_p^{r-1}(P) \otimes Z) \subset B_p^{r-1}(\Lambda)$.

Clearly all three follow from (2.7), (2.8), and the fact that α^r is induced from the identity map $C(P) \otimes C(\Omega) \to \Lambda$.
Now by passing to quotients we have an induced map

$$\alpha^r : E^r(P) \otimes H(\Omega) \to E^r(\Lambda).$$

Since the original α^r is induced from the identity map of $C(P) \otimes C(\Omega) \to \Lambda$, we have $\alpha^r \circ (d^r \otimes 1) = d^r \circ \alpha^r$.

(4.29) Definition. We put

$$\pi^o = \zeta^o \circ \alpha^o : E^o(P) \otimes C(\Omega) \to E^o(P) ,$$

$$\pi^r = \zeta^r \circ \alpha^r : E^r(P) \otimes H(\Omega) \to E^r(P) \quad , \quad r \geq 1,$$

and we write $z * v = \pi^r(z \otimes v)$, $r \geq 0$.

Since ζ^r and α^r commute with the differentials, we have π^r commuting with the differentials, or

(4.30) $d^r(z * v) = d^r z * v$, $r \geq 1$, $z \in E^r(P)$, $v \in H(\Omega)$.

$d^o(z * v) = d^o z * v + w(z) * dv$, $w(z) = (-1)^p z$, $z \in E^o(P)$, $v \in C(\Omega)$.

Since the $*$ operation is derived from the original $\otimes$ operation in $C(P) \otimes C(\Omega)$ by passage to quotients, we have

$$\begin{aligned} &(4.31) \quad [z * v] = [z] * v \quad , \quad z \text{ cycle of } E^r(P),\ r \geq 1,\ v \in H(\Omega), \\ & \quad [z * v] = [z] * [v] \quad , \quad z \text{ cycle of } E^o(P),\ v \in Z(\Omega). \end{aligned}$$

By Proposition (4.19) the multiplication m is homotopy associative.

(4.32) Proposition. The $*$ operation in $H(\Omega)$ is associative.

Proof. We use the same notations and symbols as used in the proof of (4.19).

For an n-cube $w = w(x_1,\dots,x_n)$ in $P \times \Omega \times \Omega$, we define an $(n+1)$-cube kw by

$$(4.33) \quad kw(x_1,\dots,x_{n+1}) = (-1)^n f_{n+1}(w(x_1,\dots,x_n)),$$

where f_{n+1} is the connecting homotopy between g_o and g_1 defined in (4.19).

Clearly (4.33) implies that if w is degenerate then so is kw, and that k raises dimension by 1.

We have

$$\begin{aligned} dkw &= \textstyle\sum_{i=1}^{n+1}(-1)^i(\lambda_i^1(-1)^n f_{n+1}w - \lambda_i^o(-1)^n f_{n+1}w) \\ &= \textstyle\sum_{i=1}^{n+1}(-1)^i(-1)^n(\lambda_i^1 f_{n+1}w - \lambda_i^o f_{n+1}w) \end{aligned}$$

and

$$\begin{aligned} kdw &= (-1)^{n-1} f_n \textstyle\sum_{i=1}^{n}(-1)^i(\lambda_i^1 w - \lambda_i^o w) \\ &= \textstyle\sum_{i=1}^{n}(-1)^i(-1^{n-1}(f_n\lambda_i^1 w - f_n\lambda_i^o w) \end{aligned}$$

so that

$$\begin{aligned} dkw + kdw = &\textstyle\sum_{i=1}^{n+1}(-1)^i(-1)^n(\lambda_i^1 f_{n+1}w - \lambda_i^o f_{n+1}w) \\ &+ \textstyle\sum_{i=1}^{n}(-1)^i(-1)^{n-1}(f_n\lambda_i^1 w - f_n\lambda_i^o w) \ . \end{aligned}$$

If we apply the definitions of f_θ and λ_i^ε , $\varepsilon = 0,1$, all the terms of the alternating sums except

$$(-1)^{n+1}(-1)^n(f_1w - f_ow)$$

disappear. We write

$$dk + kd = g_o - g_1 ,$$

since f_θ is the connecting homotopy between g_o and g_1 (see the proof of (4.19).

Hence we have a chain homotopy between g_o and g_1 , and the induced maps

$$g_{o*} , g_{1*} : H(P) \otimes H(\Omega) \otimes H(\Omega) \to H(P)$$

are identical.

From (4.16), $\Omega \times \Omega \times \Omega$ is carried into Ω by f_θ ; hence the $*$ operation in $H(\Omega)$ is associative.

It follows immediately from (4.18) that the zero homology class, defined and denoted by e is the unit for $H(\Omega)$.

We now consider the chain equivalence

$$\Psi : C(X) \otimes C(\Omega) \to E^o(P)$$

due to Serre [22], which we defined by (3.19). We also recall that Ψ commutes with the differentials when we take $C(X)$ with $d_x = 0$.

(4.34) <u>Definition</u>. We define two maps κ_1 and κ_2

$$\kappa_1 , \kappa_2 : C(X) \otimes C(\Omega) \otimes C(\Omega) \to E^o(P).$$

by $\kappa_1 = \Psi \circ (1 \otimes \zeta)$ and $\kappa_2 = \pi^o \circ (\Psi \otimes 1)$,

where ζ is defined from (4.13) and π^o by (4.29).

Since Ψ, ζ, and π^o are chain maps, so are both κ_1 and κ_2 .

We understand definition (4.34) in the following manner: For a given p-dimensional cube u of X and q,q' dimensional cubes v,w of Ω, we may form cubes $K(u,v*w)$ and $K(u,v)*w$. Now (3.18) and (4.15) imply that if u,v, or w belongs to T^P or is degenerate, then so are both $K(u,v*w)$ and $\cancel{K(}u,v)*w$. Hence we have the induced maps

$$\kappa_1, \kappa_2 : C(X) \otimes C(\Omega) \otimes C(\Omega) \to E^o(P).$$

(4.35) Lemma. $\kappa_1 = \Psi \circ \phi \circ \kappa_2$

Proof. From (3.16)

$$\phi(K(u,v)*w) = B(K(u,v)*w) \otimes F(K(u,v)*w),$$

while from (3.18), property 1,

$$B(K(u,v)*w) \otimes F(K(u,v)*w) = u \otimes v*w .$$

If we apply Ψ, we have from (3.19)

$$\Psi(u \otimes v*w) = K(u,v*w).$$

Hence $\kappa_1 = \Psi \circ \phi \circ \kappa_2$.

(4.36) Lemma. For $x \in C(X)$, $u,v \in Z(\Omega)$, we have

$$\Psi(x \otimes u)*v \sim \Psi(x \otimes (u*v)) \text{ in } E^o(P).$$

Proof. We want to show that in $E^o(P)$,

$$\Psi(x \otimes u)*v \sim \Psi(x \otimes (u*v)),$$

i.e. $\pi^o(\Psi(x \otimes u) \otimes v) \sim \Psi(x \otimes (u*v))$,

i.e. $\pi^o \circ (\Psi \otimes 1)(x \otimes u \otimes v) \sim \Psi(1 \otimes \zeta)(x \otimes u \otimes v)$,

i.e. $\kappa_1(x \otimes u \otimes v) \sim \kappa_2(x \otimes u \otimes v)$.

Now if there were a chain homotopy between κ_1 and κ_2, we would have our result.

We consider the operator k defined by (3.22) and put $s = k\kappa_2$. Now

$$d^o s + sd = d^o k\kappa_2 + k\kappa_2 d$$

$= d^o k\kappa_2 + kd^o\kappa_2$, since π^o and Ψ commute with the differentials.

$= (d^o k + kd^o)\kappa_2$

$= (1 - \Psi \cdot \phi)\kappa_2$, from the proof of (3.17), see page 36

$= \kappa_2 - \Psi \cdot \phi \cdot \kappa_2$

$= \kappa_2 - \kappa_1$, from lemma (4.35).

Hence a chain homotopy exists between κ_1 and κ_2 so that, if x is a chain of X, and u,v are cycles of Ω , then $\kappa_2(x \otimes u \otimes v)$ and $\kappa_1(x \otimes u \otimes v)$ are homologous.

Using (4.36) and the fact that $*$ commutes with passage to homology, we have, going to E^1 ,

(4.37) $\qquad (x \otimes u) * v = x \otimes (u * v) \quad$ in $E^1(P)$,

where $x \in C(X)$, $u,v \in H(\Omega)$.

(4.38) Proposition. In $E^r(P)$, we have

$$z * (v * w) = (z * v) * w ,$$

whenever $z \in E^r(P)$, $r \geq 1$, $v,w \in H(\Omega)$.

Proof. Now (3.23) implies that $E^1(P)$ is identified with $C(X) \otimes H(\Omega)$ by the induced map Ψ_*, so that

$$(x \otimes u) * v = \Psi_*^{-1}(\Psi_*(x \otimes u) * v) \quad \text{for } x \in (C(X), u,v \in H(\Omega).$$

From (3.33) we see that the differential d^1 of E^1 is transformed into the natural boundary operator $d_x \otimes 1$; while from (3.34) we see that E^2 is the homology group of $C(X) \otimes H(\Omega)$ by $d_x \otimes 1$.

Taking $x \in Z(X)$, $u,v \in H(\Omega)$ and using the natural embedding of $H(X) \otimes H(\Omega)$ into $E^2 = H(C(X) \otimes H(\Omega))$, we have

$$[x] \otimes (u * v) = [x \otimes (u * v)]$$
$$= [(x \otimes u) * v] \quad , \text{ from } (4.37)$$
$$= [x \otimes u] * v = ([x] \otimes u) * v \quad , \text{ since both } * \text{ and } \otimes \text{ commute with passage to homology.}$$

Now we see from (4.32) that the $*$ operator is associative on $H(\Omega)$; consequently (4.37) gives

(4.39) $$(x \otimes u) * (v * w) = ((x \otimes u) * v) * w \quad \text{in } E^1(P),$$

whenever $x \in C(X)$, $u,v,w \in H(\Omega)$. In view of (3.23), expressions of the form $x \otimes u$ span E^1 ; hence, passing to homology we have, for $z \in E^r(P)$, $r \geq 1$, $v,w \in H(\Omega)$,

$$z * (v * w) = (z * v) * w.$$

(4.40) Proposition. In $E^r(P)$ we have

$$z * e = z \quad , \text{ for } z \in E^r(P), \ e \text{ unit of } H(\Omega).$$

Proof. We take e as the unit of $H(\Omega)$.

Replacing v by e in (4.37),

$$(x \otimes u) * e = x \otimes (u * e) = x \otimes u \quad \text{in } E^1(P),$$

with $x \in C(X)$, $u,e \in H(\Omega)$. Again $x \otimes u$ span E^1 and passing to homology we have

$$z * e = z \qquad \text{for } z \in E^r(P), \ e \text{ unit of } H(\Omega).$$

It is clear from (4.3) that the map μ of (4.2) may be written in detail as

$$\mu : Q_p(X) \otimes Q_q(Y) \to Q_{p+q}(X \times Y).$$

It is also clear from (4.9) that the map $m^\#$ of (4.12) may be written in detail as

$$m^\# : Q_{p+q}(P \times \Omega) \to Q_{p+q}(P).$$

Now combining both, we see that our ζ of (4.13) may be written as

$$\zeta : Q_p(P) \otimes Q_q(\Omega) \to Q_{p+q}(P),$$

and consequently our induced homomorphism ζ_* may be written as

(4.41) $$\zeta_* : H_p(P) \otimes H_q(\Omega) \to H_{p+q}(P).$$

We recall that our chain equivalence Ψ may be written in detail as

$$\Psi : C_p(X) \otimes C_q(\Omega) \to E^o_{p,q}(P) ,$$

so that $\kappa_1 (= \Psi \circ (1 \times \zeta))$, and consequently κ_2, may be written as

(4.42) $$\kappa_1, \kappa_2 : C_p(X) \otimes C_q(\Omega) \otimes C_n(\Omega) \to E^o_{p,q+n}(P).$$

If we again apply Ψ , we have a map

(4.43) $$E^o_{p,q}(P) \otimes C_n(\Omega) \to E^o_{p,q+n}(P).$$

Now by using the methods described earlier in this chapter, we have from (4.42), by passing to homology, the maps

(4.44) $$E^r_{p,q}(P) \otimes H_n(\Omega) \to E^r_{p,q+n}(P) \quad , \quad r \geq 1.$$

Together, the results of this chapter have proven the following theorem of Bott and Samelson [3].

(4.45) <u>Theorem</u>. For a 1-connected space X, the map $m : P \times \Omega \to P$ induces the following homomorphisms

(1) $H_p(\Omega) \otimes H_q(\Omega) \to H_{p+q}(\Omega)$.

(2) $E^o_{p,q}(P) \otimes C_n(\Omega) \to E^o_{p,q+n}(P)$,

(3) $E^r_{p,q}(P) \otimes H_n(\Omega) \to E^r_{p,q+n}(P)$, $r \geq 1$,

where the pairing is written $*$ and has the following properties:

(1) The pairing is bilinear, and associative, that is, for $x \in E^r$, $r \geq 1$; $u,v,w \in H(\Omega)$; the relations

$$(x * u) * v = x * (u * v) \quad \text{and} \quad (u * v) * w = u * (v * w)$$

hold; e satisfies

$$e * v = v * e = v \quad , \quad \text{for} \quad v \in H(\Omega) ,$$

and

$$x * e = x , \qquad \text{for} \quad x \in E^r(P) \quad , \quad r \geq 1.$$

(2) $*$ commutes with the differentials

(3) $*$ commutes with the passage to homology.

(4) In E^1 and E^2 we have, for $x \in C(X)$, respectively $H(X)$, $u,v \in H(\Omega)$.

$$(x \otimes u) * v = x \otimes (u * v) ,$$

where E^1 is identified with $C(X) \otimes H(\Omega)$, and
where $H(X) \otimes H(\Omega)$ is imbedded into $E^2 = H(X,H(\Omega))$.
We take the coefficients for the groups $C(\Omega)$, $H(\Omega)$, $C(X)$, $H(X)$ and E^r from the group of integers.

We again note here that $H(\Omega)$, with $*$ as multiplication, is called the Pontryagin algebra $H_*(\Omega)$ of Ω .

(4.46) Remark. Bott and Samelson [3] have also generalized this theorem and T. Kudo [16] has given a similar theorem in a general form.

CHAPTER 5

Theorem A

We are now ready to investigate the first of our two major theorems and its corollaries. We follow the methods of [3].

As in the previous chapter, P is the space of paths in a 1- connected space X, ending at x_o, considered as a fiber space over X, with fiber Ω, the space of loops in X, at x_o.

(5.1) Definition. An element x of $H_p(X)$, $p > 0$, is transgressive if

$$d^2x = d^3x = \dots = d^{p-1}x = 0.$$

(5.2) Claim. This definition implies (3.35) and (3.36).

Proof of Claim. Since (3.35) and (3.36) are equivalent by (3.38), it is sufficient to show that (5.1) implies (3.35). To do this we must show that a certain subgroup of $H_p(X)$ is mapped into a factor group of $H_{p-1}(\Omega)$ by the differential d^p.

Taking e as the generator of H_o of any space, given by a point, we may identify $H(X)$ with $H(X) \otimes e$ and $H(\Omega)$ with $e \otimes H(\Omega)$ in E^2. From (4.45) these identifications are compatible with the $*$ operation. Now d^2x represents $d^2(x \otimes e)$ or $d^2(e \otimes x)$.

Since d^2 is trivial, any $x \in E^2$ is a cycle of E^2 and determines an element of E^3, namely $[x]$, its homology class. In E^3, we again denote $[x]$ by x and repeat the process, so that eventually, any element $x \in E^r$ is a cycle for E^r, $r = 2,\dots,p - 1$. Hence, the element d^px belongs to a certain factor group of $H_{p-1}(\Omega)$, since the differential d^p maps $E^p_{p,o}$ ($\subset E^2_{p,o} \approx H_p(X) \otimes e$) into $E^p_{o,p-1}$.

We again state Theorem A.

(5.3) Theorem A. We take $H(X)$, with coefficients in the principal ideal domain R, as being R-free and all elements of $H(X)$ as being transgressive (in P).

Now the Pontryagin algebra $H_*(\Omega)$ is the free associative algebra, with unit e, generated by a subgroup of $H(\Omega)$ which is the isomorphic image of the positive dimensional elements of $H(X)$ under a map reducing dimension by one.

Proof of Theorem A. The proof follows that of [3].

Since $H(X)$ is R-free, theorem (3.34) gives

$$E^2(P) = H(X;H(\Omega)) = H(X) \otimes H(\Omega).$$

(5.4) Definition. If, for any element z of E^2, $z \in E^2_{p,q}$, all the differentials $d^2z,\ldots,d^{p-1}z$ are trivial, E^2 is said to be totally transgressive.

(5.5) Claim. We claim $E^2(P)$ to be totally transgressive.

Proof of Claim:

From (4.45),

$$d^2(x \otimes v) = d^2((x \otimes e) * v) = d^2(x * v) = (d^2x) * v$$

and

$$[x * v] = [x] * v = x * v \quad \text{in} \quad E^3(P) ,$$

so that, for $z = (x \otimes v) \in E^2(P)$,

$$d^2z = d^2(x \otimes v) = (d^2x) * v.$$

Since $H(X)$ is transgressive, $d^2x = 0$, and, subsequently, $d^2z = (d^2x) * v = 0$.

Similarly, from (4.45), for $r > 2$,

$$d^r(x \otimes v) = d^r(x) * v \quad \text{and} \quad [x * v] = x * v \quad \text{in} \quad E^{r+1},$$

so that, for $2 \leq r \leq p - 1$,

$$d^r z = (d^r x) * v = 0$$

from the transgressivity of $H(X)$ and the claim is proven.

Our argument for proving (5.3) follows from the fact that, since P is contractible[5], $H(P)$ and $E^\infty(P)$ can contain no non trivial elements and that, therefore, the elements of $E^2(P)$ must be killed under the differentials d^r.

(5.6) Definition. For a space Y, we denote by $H_+(Y)$ the elements of $H(Y)$ of positive dimension.

We choose a base

(5.7) $B = \{x_i\}$, $i \in J$, an index set, of $H_+(X)$, consisting of homogeneous-dimensional elements.

(5.8) Definition. For each $x_i \in B$, we define an element x_i' of $H_+(\Omega)$ by

$$d^p x_i = x_i' \quad , \quad \text{with } p = \text{dimension of } x_i .$$

This definition is possible because, for $x_i \in H_p(X)$, $d^p x_i \in H_{p-1}(\Omega)$, since $H(X)$ is transgressive.

(5.9) Definition. We define an element $x'_{i_1 i_2 \ldots i_k}$ of $H(\Omega)$ by

$$x'_{i_1 i_2 \ldots i_k} = x'_{i_1} * x'_{i_2} * \ldots * x'_{i_k} ,$$

where $k \geq 1$, $i_j \in J$, $j = 1,\ldots,k$. The length of $x'_{i_1 \ldots i_k}$ is defined to

[5] Well known fact which may be found in [23], and others.

be k and the <u>height</u> of $x'_{i_1 \ldots i_k}$ is defined to be the dimension of x'_{i_1}.

We denote the collection of elements $x'_{i_1 \ldots i_k}$ by M.

Should the elements of M be independent and, together with e, form a basis for $H(\Omega)$ our theorem would obviously be proven; the required subgroups of $H(\Omega)$ being generated by the x'_i.

(5.10) <u>Lemma</u>. The elements of M are linearly independent.

<u>Proof of Lemma</u>. We shall assume that there are linear relations between the elements of M and we shall see that this assumption gives rise to a contradiction.

We examine those linear relations in which the maximum length occuring is as small as possible, say ℓ. Take such a relation

$$r \equiv \sum c_\alpha z_\alpha = 0,$$

where $z_\alpha \in M$ and has length $\leq \ell$.

Since r is a linear relation, we may write it as a sum $r = r_1 + r_2$, where r_1 contains all the elements of maximum height p. Since $z_\alpha \in M$, we see from (5.9) that z_α may be written as $x'_\alpha * \bar{z}_\alpha$, where x'_α is defined by (5.8) and (5.9) and where $\bar{z}_\alpha$ belongs to M or is equal to e. Now $x_\alpha \in H(X)$ and $\bar{z}_\alpha \in M \subset H(\Omega)$, so that we may form elements of the form

$$y_\alpha = x_\alpha \otimes \bar{z}_\alpha \in E^2(P).$$

We put $Y = \sum c_\alpha y_\alpha = Y_1 + Y_2$, where Y_1 and Y_2 correspond to r_1 and r_2 respectively.

We now consider the case in $E^{p+1}(P)$. Since $d^q(x_\alpha \otimes \bar{z}_\alpha) = (d^q x_\alpha) * \bar{z}_\alpha$ from (4.45) and $(d^q x_\alpha) * \bar{z}_\alpha = x'_\alpha * \bar{z}_\alpha$ from (5.8), the elements $x_\alpha \otimes \bar{z}_\alpha$ in Y_2 are mapped onto the $x'_\alpha * \bar{z}_\alpha$ in r_2 by the appropriate differentials d^q. By (5.9) the height of $x'_\alpha * \bar{z}_\alpha$ would be the dimension of x'_α ; so that, considering the facts that r_1 contains all the elements of maximum height p and that d^q reduces dimension by one, we have $q \leq p$. Total transgressivity implies that the d^q, $q \leq p$, are all trivial; hence r_2 is 0 in E^{P+1}.

$r_1 + r_2 = r = 0$ implies r_1 is also 0 in E^{P+1}. Since $d^q(x_\alpha \otimes \bar{z}_\alpha) = (d^q x_\alpha) * \bar{z}_\alpha$ from (4.45), $(d^q x_\alpha) * \bar{z}_\alpha = x'_\alpha * \bar{z}_\alpha$ from (5.8), and r_1 contains the elements of maximum height p, the elements of Y_1 are mapped onto r_1 by the differential d^{P+1}. Hence , $d^{P+1}(Y_1) = 0$. Because of total transgressivity we have $d^q = 0$, $q \leq p$, so Y_1 is not image under any d^q, $q \leq p$; therefore, $Y_1 = 0$, or else $E^\infty(P)$ would contain a non trivial element.

From (5.9), we have that, since the $z_\alpha = x'_\alpha * \bar{z}_\alpha$ are of maximum length $\leq \ell$, the $\bar{z}_\alpha$, occuring in Y_1, are of length $< \ell$. Then, because of the minimality assumption on ℓ, the $\bar{z}_\alpha$ form a free subgroup of $H(\Omega)$. In addition to this free subgroup of $H(\Omega)$, we have assumed $H(X)$ free (in the statement of Theorem A); therefore, the elements $x_\alpha \otimes \bar{z}_\alpha$, occuring in Y_1, are independent in E^2, and, because of total transgressivity, the elements $x_\alpha \otimes \bar{z}_\alpha$ are also independent in E^{P+1}. However, we have already shown that Y_1 is 0 in E^{P+1}; hence we have a contradiction and have proven that the elements of M are linearly independent.

(5.11) Lemma. The elements of M, together with e, form a basis for $H(\Omega)$.

Proof of the Lemma. We again prove by contradiction.

Let us assume that there are elements in $H_+(\Omega)$ which are not linear combinations of elements of M. Let n be the smallest dimension in which this happens.

All elements of $E^2_{n-q,q}$ are of the form

$$\sum x_\alpha \otimes t_\alpha ,$$

$q < n$, with dimension $t_\alpha = q < n$; the t_α being generated by M and e.

We consider the action of the differentials $d^2,\ldots,d^{n+1}$, where

$$d^{q+1} : E^{q+1}_{n+1,0} \to E^{q+1}_{n-q,q} ,$$

$2 \leq q + 1 \leq n + 1$.

Recalling that, whenever dimension $x = p$, $d^p(x \otimes t) = ((d^p x) * t)$ modulo the images of $d^2,\ldots,d^{p+1}$ = $(x' * t)$ modulo the images of $d^2,\ldots,d^{p-1}$, then $E^{n+2}_{0,n}$ is a quotient group of $H_n(\Omega)$ by a subgroup which is contained in a subgroup generated by M, since $d^{n+1}(E^{n+1}_{0,n}) = 0$ from (2.25)[6]. However, from (2.27)[6],

$$E^{n+2}_{0,n} = E^{n+3}_{0,n} = \ldots = E^{\infty}_{0,n} ,$$

and, since P is contractible, there can be no non trivial element in $E^{\infty}(P)$; hence $E^{n+2}_{n,0} = 0$.

We have a contradiction to the assumption that there are elements in $H_+(\Omega)$ which are not linear combinations of elements of M; hence the Lemma is proven.

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[6]We recall from Chapter 3 that the filtration on $C(P)$ is regular.

We return to the proof of (5.3).

From lemmas (5.10) and (5.11), the element $x'_{i_1 i_2 \ldots i_k}$ of M and e form a basis for $H(\Omega)$. Clearly the algebraic structure follows from the relation

$$x'_{i_1 \ldots i_k} * x'_{j_1 \ldots j_\ell} = x'_{i_1 \ldots i_k j_1 \ldots j_\ell}$$

and clearly the required map, reducing dimension by one, is the differential d^P.

The following well known result is a corollary to this theorem.

(5.12) Corollary. The Pontryagin algebra of the space Ω of loops in the space S^n, $n > 1$, is the free associative algebra on one generator of dimension $n - 1$.

Proof. From lemma (3.42) all spherical homology cycles are transgressive, so that we may apply Theorem A and the result follows.

Generalizing corollary (5.12), we have

(5.13) Corollary. The Pontryagin algebra of the space Ω of loops in the one point union of k spheres of dimension $n_i > 1$, $i = 1,\ldots,k$, is the free associative algebra on k generators of dimension $n_i - 1$.

Proof. Identical to that of (5.12).

This corollary due to [3], has been very important in homotopy theory, being one of the results required by Hilton [9] in his determining of the homotopy groups of the one point union of k spheres of dimension $n_i > 1$, $i = 1,\ldots,k$.

We give the usual definition of the suspension of a space[7].

(5.14) Definition. The suspension of a space X, denoted by SX, is defined to be the quotient space of $X \times I$ in which $X \times 0$ is identified with some point x_o, and $X \times 1$ is identified with some point x_1.

(5.15) Proposition. If SX is the suspension of a 0-connected space X, then, in the spectral sequence of P, the space of paths over SX, all elements of $H_+(SX)$ are transgressive.

Proof. Clearly from (5.14), SX is 1-connected.

We put

$$X_o = \{(x,t) \mid x \in X , 0 < t \leq \tfrac{1}{2}\} \cup \{x_o\} ,$$
$$X_1 = \{(x,t) \mid x \in X , \tfrac{1}{2} \leq t < 1\} \cup \{x_1\}$$

and identify each $x \in X$ with $(x, \frac{1}{2}) \in SX$.

We define a map $f : (X_o,X) \to (SX,X_1)$ by

$$\begin{cases} f(x_o) = x_o \\ f(x,t) = (x,2t) , & 0 < t < \frac{1}{2} \\ f(x) = x_1 , & x \in X. \end{cases}$$

Clearly f is homotopic to the inclusion map

$$(X_o,X) \subset (SX,X_1).$$

We may also consider f as a map from (X_o,X) to (SX,x_1).

Since $X_o - X = SX - X_1$ from the definitions, the inclusion $(X_o,X) \subset (SX,X_1)$ is an excision map[8].

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[7]See [23], page 41.

[8]For a definition of excision see [23], page 187.

From [23], page 188, Theorem 1, an excision map induces an isomorphism in homology, so that

$$H(X_o,X) \approx H(SX,X_1).$$

Because we have defined X_1 in such a way as to be contractible, we have

$$H(SX,X_1) \approx H(SX,x_1).$$

Thus

$$f_* : H(X_o,X) \to H(SX,x_1)$$

is an isomorphism.

Recalling that we have an inclusion $X_o \subset SX$, we can find a map $g : X_o \to P$, such that $p \circ g = f$, since X_o is defined in such a way as to be contractible. Now, if we let Ω be the fiber of P over x_1, g maps the subset $X \subset X_o$ into Ω and so defines a pair map

$$g : (X_o,X) \to (P,\Omega).$$

Clearly

$$f_* = p_* \circ g_* : H(X_o,X) \to H(SX,x_1) ,$$

where p_* maps $H(P,\Omega)$ onto $H(SX,x_1)$.

Now the result follows from (3.36), the definition of transgression.

(5.16) <u>Corollary</u>. The Pontryagin algebra of the space Ω of loops in SX, the suspension of a 0-connected space X, is a free associative algebra whose generators can be determined from a knowledge of $H_+(SX)$.

<u>Proof</u>. Immediate, since (5.15) implies that we may apply Theorem A.

CHAPTER 6

Theorem B

Lastly we study a relationship between a product in homotopy and the Pontryagin product in homology. This homotopy product is the Whitehead product, which was introduced by J.H.C. Whitehead [30], in 1941, and which has been redefined and generalized many times since.

We take P_X to be the space of paths in a 1-connected space X, ending at x_o, considered as a fiber space over X, with fiber Ω_X, the space of loops in X, at x_o.

(6.1) <u>Definition</u>. For a given map $f : (I^n, \dot{I}^n) \to (X, x_o)$, we define a map $T_i f : I^{n-1} \to \Omega_X$ by

$$T_i f(x_1, \ldots, x_{n-1})(t) = f(x_1, \ldots, x_{i-1}, t, x_i, \ldots, x_{n-1}),$$

where $i, n \in \mathbb{Z}$, $1 < i \leq n$.

It is obvious from (6.1) that the operator T_i induces an isomorphism

$$\text{(6.2)} \qquad T_i : \pi_n(X, x_o) = \pi_n(X) \to \pi_{n-1}(\Omega_X), \quad 1 < i \leq n,$$

where the basejoint of Ω_X is the degenerate loop e, at x_o, defined by (4.11). If we replace addition in $\pi_{n-1}(\Omega_X)$ by the multiplication in Ω_X, as defined by (4.9), we also have an isomorphism when $i = 1$.

We take another 1-connected space $Y \subset X$, such that $x_o \in Y$ and denote by $P_Y(\Omega_Y)$ the space of paths (respectively, loops) in Y ; obviously $P_Y \subset P_X$. Clearly T_i induces an isomorphism

$$\text{(6.3)} \qquad T_i : \pi_n(Y) \to \pi_{n-1}(\Omega_Y).$$

The exactness of the homotopy sequences of the pairs (X,Y) and (Ω_X,Ω_Y), together with (6.2) and (6.3), allows the application of the "five lemma" (see [13], for a definition); hence, the operator T_i also defines an isomorphism

(6.4) $$T_i : \pi_n(X,Y) \to \pi_{n-1}(\Omega_X,\Omega_Y) \quad , \quad i < n,$$

and commutes with the boundary operator. Now $T_o = T_2$ ($T_o = T_2 = -T_1$ for the group $\pi_2(X,Y)$) maps the homotopy sequence of (X,Y) isomorphically onto that of (Ω_X,Ω_Y) with degree -1. We may replace T_2 by $-T_1$ because, since the interchange of two axis is an orientation reversing homeomorphism of I^n ,

(6.5) $$T_i = (-1)^{i+j}T_j \quad , \quad i \leq i,j \leq n.$$

(6.6) <u>Definition</u>. We define an isomorphism T from $\pi_n(X)$ to $\pi_{n-1}(\Omega_X)$ by putting

$$T = d \circ p_X^{-1} : \pi_n(X) \to \pi_{n-1}(\Omega_X) \ ,$$

where $p_X : \pi_n(P_X,\Omega_X) \to \pi_n(X)$ is the isomorphism induced from the projection $p_X : P_X \to X$ and where d, the boundary operator : $\pi_n(P_X,\Omega_X) \to \pi_{n-1}(\Omega_X)$, is an isomorphism because $\pi_n(P_X) = 0$ in the homotopy sequence of (P_X,Ω_X), since P_X is contractible.

Samelson [20], page 747, has shown that T , applied to $\pi_n(X)$, is the same as the isomorphism T_n defined above.

We note that the isomorphisms from $\pi_n(X)$, respectively $\pi_n(Y)$, to $\pi_{n-1}(\Omega_X)$, respectively $\pi_{n-1}(\Omega_Y)$, may be given by $(-1)^n d \circ p_X^{-1}$, respectively $(-1)^n d \circ p_Y^{-1}$.

(6.7) <u>Definition</u>. We define a map τ between $\pi_n(X)$ and $H_{n-1}(\Omega_X)$ by

$$\tau = h \circ T : \pi_n(X) \to H_{n-1}(\Omega_X) ,$$

where T is defined above and where h is the Hurewicz homomorphism.

(6.8) <u>Definition</u>. For $\alpha \in \pi_p(X)$, $\beta \in \pi_q(X)$, we define <u>the Whitehead product of α and β</u> to be the element

$$[\alpha,\beta] = (\nabla \circ \alpha \vee \beta \circ w) \in \pi_{p+q-1}(X),$$

where

$\nabla : X \vee X \to X$ is the folding map,

$\alpha \vee \beta : S^p \vee S^q \to X \vee X$ is induced from

$$\alpha : S^p \to X \quad \text{and} \quad \beta : S^q \to X \quad ,$$

$w : S^{p+q-1} \to S^p \vee S^q$ is the attaching map used to attach the boundary of a $p + q$ - cell e^{p+q} to $S^p \vee S^q$ $(= e^0 \cup_c e^p \cup_c e^q)$ in the definition of S^{p+q} as a cell complex[9].

Clearly the Whitehead product depends only on the homotopy class and not on the representative of these classes.

We again state Theorem B.

(6.9) <u>Theorem B</u>. If, for a 1-connected space X, we have $\alpha \in \pi_{p+1}(X)$, $\beta \in \pi_{q+1}(X)$, $p,q \geq 1$, then

$$\tau[\alpha,\beta] = \pm (\tau\alpha * \tau\beta - (-1)^{pq}\tau\beta * \tau\alpha) ,$$

where the map τ is defined by (6.7), the operation [,] is defined by (6.8), and the operation $*$ is defined by (4.15).

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[9]A definition of cell complexes and attaching maps may be found in [23] and others.

Proof of Theorem B.

The proof follows that given by [20].

1st Proof.

(6.10) Definition. We put $X = S^{p+1} \times S^{q+1}$ and $Y = S^{p+1} \vee S^{q+1}$.

(6.11) Lemma. For X and Y defined by (6.10) we have

$$\pi_i(\Omega_X,\Omega_Y) \approx H_i(\Omega_X,\Omega_Y) \approx \begin{cases} 0, & i < p+q+1 \\ \mathbb{Z}, & i = p+q+1 \end{cases}$$

Proof of Lemma.

We recall that

(6.12) $$H_i(Y) = H_i(S^{p+1} \vee S^{q+1}) \approx \begin{cases} \mathbb{Z}, & i = 0,\ p+1,\ q+1 \\ 0, & \text{otherwise} \end{cases} ;$$

so that, applying the Künneth formula to $H_i(X)$, we have

(6.13) $$H_i(X) = H_i(S^{p+1} \times S^{q+1}) \approx \begin{cases} \mathbb{Z}, & i = 0,\ p+1,\ q+1,\ p+q+2 \\ 0, & i \leq p+q+1,\ i \neq 0,\ p+1,\ q+1 \end{cases} ,$$

and

(6.14) $$H_i(X) \approx H_i(Y) \quad \text{for} \quad i < p+q+2.$$

In view of (6.12) and (6.14), the exactness of the homology sequence of the pair (X,Y) implies

(6.15) $$H_i(X,Y) \approx \begin{cases} 0, & i \leq p+q+1 \\ \mathbb{Z}, & i = p+q+2. \end{cases}$$

Now using the relative Hurewicz theorem[10] , we have

(6.16) $$\pi_i(X,Y) \approx \begin{cases} 0, & i \leq p+q+1 \\ \mathbb{Z}, & i = p+q+2. \end{cases}$$

Applying the isomorphism T_o (as defined earlier), it follows that

(6.17) $$\pi_i(\Omega_X,\Omega_Y) \approx \begin{cases} 0, & i < p+q+1 \\ \mathbb{Z}, & i = p+q+1 \end{cases}$$

and, by the relative Hurewicz theorem, that

(6.18) $$H_i(\Omega_X,\Omega_Y) \approx \pi_i(\Omega_X,\Omega_Y) \quad , \quad i = p+q+1.$$

In the case where p or $q = 1$, the result still holds, since the H-space Ω_X is simple ([22], page 479).

Returning to the proof of Theorem B, we determine the image of the homomorphism

$$d : \pi_{p+q+1}(\Omega_X,\Omega_Y) \to \pi_{p+q}(\Omega_Y)$$

in the homotopy sequence of the pair (Ω_X,Ω_Y).

In the relative homotopy sequence of the pair (X,Y) the map $\pi_n(Y) \to \pi_n(X)$, for all n, is an epimorphism; hence, by exactness,

(6.19) $$d(\pi_{p+q+2}(X,Y)) \approx \pi_{p+q+1}(Y) \ ,$$

or, in other words,

(6.20) $$\text{kernel}(\pi_{p+q+1}(Y) \to \pi_{p+q+1}(X)) \approx \mathbb{Z},$$

since $\pi_{p+q+2}(X,Y) \approx \mathbb{Z}$.

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[10]For a statement of the relative Hurewicz Theorem see Theorem 4 on page 397 of [23].

(6.21) If $\alpha_o \in \pi_{p+1}(S^{p+1})$ and $\beta_o \in \pi_{q+1}(S^{q+1})$ are defined from the identity maps on S^{p+1} and S^{q+1}, we denote by α', respectively β', the images of α_o, respectively β_o, under inclusion in Y.

Clearly $\alpha' \in \pi_{p+1}(Y)$ and $\beta' \in \pi_{q+1}(Y)$.

Because of the way we have defined α' and β', it follows from definition (6.8) that

(6.22) $[\alpha',\beta']$ generates the kernel of the map $\pi_{p+q+1}(Y) \to \pi_{p+q+1}(X)$, that is,

$d(\pi_{p+q+2}(X,Y))$ is the infinite cyclic group generated by $[\alpha',\beta']$; so that,

(6.23) $d(\pi_{p+q+1}(\Omega_X,\Omega_Y))$ is the infinite cyclic group generated by $T[\alpha',\beta']$.

(6.24) Lemma. $H_*(\Omega_X)$, the Pontryagin algebra of Ω_X, with unit e, is generated by $\tau\alpha_o$ and $\tau\beta_o$ subject to the relation

$$\tau\alpha_o * \tau\beta_o = (-1)^{pq}\tau\beta_o * \tau\alpha_o ,$$

where the operation $*$ is defined by (4.15).

Proof of the Lemma.

We let Ω_1, respectively Ω_2, be the space of loops over S^{p+1}, respectively S^{q+1}. From (5.12), $H_*(\Omega_1)$, respectively $H_*(\Omega_2)$, is the free associative polynomial algebra in the variable $\tau\alpha_o$ of dimension p, respectively in the variable $\tau\beta_o$ of dimension q.

Since, by (4.7), $\Omega_X = \{f : I \to X(= S^{p+1} \times S^{q+1}) | f(0) = f(1) = x_o = s_1 \times s_2\}$ and $\Omega_1 \times \Omega_2 = \{f_1 : I \to S^{p+1} | f_1(0) = f_1(1) = s_1\} \times \{f_2 : I \to S^{q+1} | f_2(0) = f_2(1) = s_2\}$ and since, by the properties of the cartesian product

$X = S^{p+1} \times S^{q+1}$, $\{f : I \to X | f(0) = f(1) = x_o)$ =

$\{f_1 : I \to S^{p+1} | f_1(0) = f_1(1) = s_1\} \times \{f_2 : I \to S^{q+1} | f_2(0) = f_2(1) = s_2\}$,

we have

(6.25) $$\Omega_X = \Omega_1 \times \Omega_2 .$$

Now, applying the Künneth formula to $H(\Omega_X) = H(\Omega_1 \times \Omega_2)$,

(6.26) $$H(\Omega_X) = H(\Omega_1) \otimes H(\Omega_2) ,$$

since $H(\Omega_1)$ and $H(\Omega_2)$ are free by (5.12). Should $x \in \Omega_1$, $y \in \Omega_2$ (or $x \in \Omega_2$, $y \in \Omega_1$) we obviously have

(6.27) $$m(x,y) \in \Omega_X ,$$

where m is defined by (4.9); so that Ω_X is the direct product of Ω_1 and Ω_2 with respect to m.

If we define a map $\lambda : \Omega_1 \times \Omega_2 \to \Omega_2 \times \Omega_1$ by

(6.28) $$\lambda(x,y) = (y,x) ,$$

it is well known that

(6.29) $$\lambda(v \otimes u) = (-1)^{pq}(u \otimes v) \quad , \quad u \in H_p(\Omega_1), \quad v \in H_q(\Omega_2).$$

Now it follows from the definitions of m and λ that we have the following commutative diagram:

(6.30)
$$\begin{array}{ccc} (\Omega_1 \times \Omega_2) \times (\Omega_1 \times \Omega_2) & \xrightarrow{\quad m \quad} & \Omega_1 \times \Omega_2 \\ {\scriptstyle 1 \times \lambda \times 1} \updownarrow \approx & & {\scriptstyle 1} \updownarrow \approx \\ (\Omega_1 \times \Omega_1) \times (\Omega_2 \times \Omega_2) & \xrightarrow{\quad m \times m \quad} & \Omega_1 \times \Omega_2 \end{array}$$

Diagram (6.30) implies that the multiplication in $H(\Omega_X) = H(\Omega_1 \times \Omega_2)$ has the property

$$(6.31) \qquad (u \otimes v) * (u' \otimes v') = (-1)^{pq}(u * v') \otimes (v * v') ;$$

clearly the Pontryagin algebra $H_*(\Omega_X)$ is the skew tensor product of the Pontryagin algebras $H_*(\Omega_1)$ and $H_*(\Omega_2)$.

Identifying $\tau\alpha_o$ with $\tau\alpha_o \otimes e$ and $\tau\beta_o$ with $e \otimes \tau\beta_o$ in $H_*(\Omega_X) = H_*(\Omega_1) \otimes H_*(\Omega_2)$, (6.31) gives

$$(6.32) \qquad \tau\alpha_o * \tau\beta_o = (-1)^{pq}\tau\beta_o * \tau\alpha_o \quad ;$$

so that, in view of (5.12), the Pontryagin algebra $H_*(\Omega_X)$ is generated by $\tau\alpha_o$ and $\tau\beta_o$ subject to (6.32).

Hence the lemma is proven.

We return to the proof of Theorem B.

We recall from (5.13) that $H_*(\Omega_Y)$, the Pontryagin algebra of Ω_Y, with unit e, is the free associative algebra on two generators $\tau\alpha'$ and $\tau\beta'$, where $\tau\alpha'$, respectively $\tau\beta'$, is the spherical homology element determined from $T\alpha'$, respectively $T\beta'$.

Since, from (4.9), the inclusion $i : \Omega_Y \to \Omega_X$ is clearly homomorphic to m, i induces a homomorphism, also called i, from the Pontryagin algebra $H_*(\Omega_Y)$ to the Pontryagin algebra $H_*(\Omega_X)$; therefore $\tau\alpha'$ and $\tau\beta'$, the generators of $H_*(\Omega_Y)$, are mapped into $\tau\alpha_o$ and $\tau\beta_o$, the generators of $H_*(\Omega_X)$. It now follows from lemma (6.24) that, in dimension $p + q$,

(6.33) The kernel of the homomorphism $i : H_*(\Omega_Y) \to H_*(\Omega_X)$ is the infinite cyclic group generated by

$$\tau\alpha' * \tau\beta' - (-1)^{pq}\tau\beta' * \tau\alpha' .$$

Because $\pi_{p+q+1}(\Omega_X,\Omega_Y) \approx H_{p+q+1}(\Omega_X,\Omega_Y)$ from (6.18), the exactness of the relative homology and the relative homotopy sequences of the pair (Ω_X,Ω_Y) imply that the Hurewicz homomorphism h must map the generator $T[\alpha',\beta']$ of the infinite cyclic group $d(\pi_{p+q+1}(\Omega_X,\Omega_Y))$ onto $\pm$ the generator $\tau\alpha' * \tau\beta' - (-1)^{pq}(\tau\beta' * \tau\alpha')$ of the infinite cyclic group $d(H_{p+q+1}(\Omega_X,\Omega_Y))$.

Hence

$$(6.34) \qquad \tau[\alpha',\beta'] = \pm\,(\tau\alpha' * \tau\beta' - (-1)^{pq}(\tau\beta' * \tau\alpha')),$$

where $\alpha' \in \pi_{p+1}(S^{p+1} \vee S^{q+1})$, $\beta' \in \pi_{q+1}(S^{p+1} \vee S^{q+1})$, and $[\alpha',\beta'] \in \pi_{p+q+1}(S^{p+1} \vee S^{q+1})$, α' and β' defined by (6.21).

Blakers and Massey [2], page 300, have shown that the space $S^{p+1} \vee S^{q+1}$ and the elements $\alpha' \in \pi_{p+1}(S^{p+1} \vee S^{q+1})$ and $\beta' \in \pi_{q+1}(S^{p+1} \vee S^{q+1})$ α' and β' defined by (6.21), is an universal example for binary homotopy constructions. Therefore, (6.34) may be extended to

$$(6.35) \qquad \tau[\alpha,\beta] = \pm\,(\tau\alpha * \tau\beta - (-1)^{pq}(\tau\beta * \tau\alpha)),$$

where $\alpha \in \pi_{p+1}(X)$, $\beta \in \pi_{q+1}(X)$, and $[\alpha,\beta] \in \pi_{p+q+1}(X)$, X being an arbitrary space.

Thus Theorem B is proven.

Second Proof of Theorem B.

Again we take X to be a 1-connected space and x_o to be a point in X.

The proof will be geometrical in nature and will require one of the older definitions of the Whitehead product, which is well known to be equivalent to (6.8).

(6.36) <u>Definition</u>. Let $\alpha : (I^{p+1}, \dot{I}^{p+1}) \to (X, x_o)$ and $\beta (I^{q+1}, \dot{I}^{q+1}) \to (X, x_o)$ represent the homotopy classes of the respective homotopy groups. Now the <u>Whitehead product $[\alpha, \beta]$ of α and β</u> is represented by the map k,

$$k : S^{p+q+1} = \dot{I}^{p+q+2} = (I^{p+1} \times \dot{I}^{q+1}) \cup (\dot{I}^{p+1} \times I^{q+1}) \to (X, x_o)$$

defined by

$$k(s,s') = \begin{cases} \alpha(s) , & s' \in \dot{I}^{q+1} \\ \beta(s') , & s \in \dot{I}^{p+1} \end{cases}$$

the base point of S^{p+q+1} being $\omega = (0,\dots,0)$.

(6.37) <u>Definition</u>. We put $I^{i+1} = I^i \times I$, $i = p,q$, so that $I^{p+q+2} = I^p \times I \times I^q \times I$ and define a subset K of $S^{p+q+1} (= \dot{I}^{p+q+2})$ by

$$K = (I^p \times I \times \dot{I}^q \times I) \cup (\dot{I}^p \times I \times I^q \times I) \cup (I^p \times 0 \times I^q \times 0).$$

If we collapse the two factors I in the first two terms, K becomes

$$(I^p \times 0 \times \dot{I}^q \times 0) \cup (\dot{I}^p \times 0 \times I^q \times 0) \cup (I^p \times 0 \times I^q \times 0)$$

and we may contract this to the point ω, since $I^p \times 0 \times I^q \times 0$ is clearly contractible. Hence K is contractible into ω.

(6.38) <u>Definition</u>. We define a map

$$\sigma : I^{p+q+1} = I^p \times I^q \times I \to S^{p+q+1}$$

in the following manner:

σ maps the interval $x \times y \times I$, in piecewise linear fashion, on that interval of the closed polygon in S^{p+q+1} ($= \dot{I}^{p+q+2}$) with vertices

$$\begin{cases} \omega \text{ and } (x,0,y,0) , & 0 \leq t \leq 1/4 \\ (x,0,y,0) \text{ and } (x,1,y,0) , & 1/4 \leq t \leq 3/8 \\ (x,1,y,0) \text{ and } (x,1,y,1) , & 3/8 \leq t \leq 1/2 \\ (x,1,y,1) \text{ and } (x,0,y,1) , & 1/2 \leq t \leq 5/8 \\ (x,0,y,1) \text{ and } (x,0,y,0) , & 5/8 \leq t \leq 3/4 \\ (x,0,y,0) \text{ and } \omega , & 3/4 \leq t \leq 1. \end{cases}$$

(6.39) Claim. $\sigma(I^{p+q+1}) \subset K$.

Proof of Claim. Case 1 : $t = 0,1$. Should $t = 0$ or 1, σ maps the point (x,y,t) to the point ω, which obviously belongs to K from (6.37).

Case 2 : $x \in \dot{I}^p$. Should $x \in I^p$, we have from (6.38) that $\sigma(x,y,t) \in \dot{I}^p \times I \times I^q \times I$, which is contained in K from (6.37).

Case 3 : $y \in \dot{I}^q$. Should $y \in \dot{I}^q$, we have from (6.38) that $\sigma(x,y,t) \in I^p \times I \times \dot{I}^q \times I$, which is contained in K from (6.37).

(6.40) Lemma. For $x \in I^p - \dot{I}^p$, $y \in I^q - \dot{I}^q$, $1/4 < t < 3/8$, σ is locally $1 : 1$ and onto in the neighbourhood of (x,y,t).

Proof of Lemma. Given a point $(x,t',y,0) \in S^{p+q+1}$, it is immediate that the point $(x,y,\frac{t'+2}{8})$ satisfies the required conditions on x,y and $t(\ t = \frac{t'+2}{8})$ and is such that $\sigma(x,y,\frac{t'+2}{8}) = (x,t',y,0)$.

Given two points (x,y,t), (x',y',t'), $(x,y,t$ satisfying the conditions of (6.40)) such that $(x,8t - 2,y,0) = (x',8t' - 2,y',0)$, it is immediate from checking the values of t that $(x,y,t) = (x',y',t')$, hence (x,y,t) is the only point of I^{p+q+1} mapped onto $(x,8t - 2,y,0)$ by σ.

Now σ is locally $1 : 1$ and onto in the neighbourhood of (x,y,t) and the lemma is proven.

(6.41) Lemma. Under the conditions of (6.40), σ has local degree $(-1)^p$.

Proof of Lemma. The result follows from the nature of σ and the natural orientations of I^{p+q+1} and S^{p+q+1}.

Together, the results (6.39), (6.40), and (6.41) prove the following lemma.

(6.42) <u>Lemma</u>. $\breve{\sigma}(\varepsilon) = (-1)^p \eta$,

where ε is the generator of $\pi_{p+q+1}(I^{p+q+1}, \dot{I}^{p+q+1})$, represented by the identity, and η is the generator of $\pi_{p+q+1}(S^{p+q+1}, K)$, represented by the identity.

(6.43) <u>Definition</u>. Letting $s_1; I^p \to S^p$ and $s_2 : I^q \to S^q$ be defined by collapsing the boundary to a point, we define a map:

$$s = s_1 \times s_2 : (I^p \times I^q, (I^p \times I^q)^{\cdot}) \to (S^p \times S^q, S^p \vee S^q)$$

to be the map induced by s_1 and s_2.

(6.44) <u>Lemma</u>. $T(k \circ \sigma)(x,y)(t) = k \circ \sigma(x,y,t)$,

where $T = T_{p+q+1}$ is defined by (6.1) and k represents the Whitehead product as defined by (6.36).

<u>Proof</u>. The proof is immediate, since

$$T_{p+q+1} f(x_1, \ldots, x_{p+q})(t) = f(x_1, \ldots, x_{p+q}, t)$$

from (6.1).

When T_{p+1} and T_{q+1} are defined by (6.1) and $\alpha \in \pi_{p+1}(X)$ is represented by $\alpha: (I^{p+1}, \dot{I}^{p+1}) \to (X, x_o)$ and $\beta \in \pi_{q+1}(X)$ is represented by $\beta : (I^{q+1}, \dot{I}^{q+1}) \to (X, x_o)$, it is clear from the definitions that the maps $T_{p+1}\alpha: I^{p+1} \to \Omega_X$ and $T_{q+1}\beta: I^{q+1} \to \Omega_X$ can be factored into $\alpha' \circ s_1$, respectively $\beta' \circ s_2$, where α' maps S^p into Ω_X and β' maps S^q into Ω_X.

(6.45) Definition. We put

$$c(x,y) = (e \cdot (\alpha'(x) \cdot \beta'(y)) \cdot ((\gamma(\alpha'(x)) \cdot \gamma(\beta'(y))) \cdot e),$$

where $\cdot$ represents the composition of loops as defined by (4.9) and $\gamma(f(x))$ is the homotopy inverse of $f(x)$ (considered as a loop in Ω_X) as defined in the proof of Proposition (4.20).

Since k represents the Whitehead product as defined by (6.36), we have that, if $x \in \dot{I}^p$, $T(k \circ \sigma)(x,y)$ depends on y only, and, if $y \in \dot{I}^q$, $T(k \circ \sigma)(x,y)$ depends on x only. Hence we have proven:

(6.46) $T(k \circ \sigma)$ may be factored in the form $c \circ s$, where c is defined by (6.45) and s is defined by (6.43).

Because e is the degenerate loop at x_o, c (as defined by (6.45)) is homotopic to the map d defined by

(6.47) $$d(x,y) = (\alpha'(x) \cdot \beta'(y)) \cdot (\gamma(\alpha'(x)) \cdot \gamma(\beta'(y))).$$

(6.48) Definition. We define a deformation f_t of S^{p+q+1}, with f_o as the identity on S^{p+q+1}, by extending to S^{p+q+1} the contraction of K over itself into ω.

Now, using methods similar to those used in the case of $T(k \circ \sigma)$, we can show that, for each t,

(6.49) $T(k \circ f_t \circ \sigma)$ may be factored in the form $c_t \circ s$, and consequently, the c_t form a homotopy of $c = c_o$.

Since f_t is a deformation of S^{p+q+1} with f_o = identity, the map $f_1 \circ \sigma$ is homotopic to $f_o \circ \sigma = \sigma$; hence

(6.50) $\quad f_1 \circ \sigma\ (\varepsilon) \ = \ (-1)^p \eta$,

because homotopies induce isomorphisms in homotopy.

From (6.48) and (6.39), we have

$$f_1 \circ \sigma(\dot{I}^{p+q+1}) = \omega ,$$

so that, in view of (6.50),

(6.51) $\quad f_1 \circ \sigma$ maps $(I^{p+q+1}, \dot{I}^{p+q+1})$ into (S^{p+q+1}, ω) with degree $(-1)^p$.

Now it follows from (6.36) that $k \circ f_1 \circ \sigma$ represents $(-1)^p[\alpha,\beta]$ and, consequently, maps ε into $(-1)^p[\alpha,\beta]$. By (6.4), $T(k \circ f_1 \circ \sigma)$ map the natural generator of $H_{p+q}(I^{p+q}, \dot{I}^{p+q})$ into $(-1)^p\tau[\alpha,\beta] \in H_{p+q}(\Omega_X,e)$. From (6.49) we have that $T(k \circ f_1 \circ \sigma)$ may be factored into $c_1 \circ s$.

Since s is known to be a relative homeomorphism, and consequently, $H_i(I^p \times I^q, (I^p \times I^q)\cdot) \approx H_i(S^p \times S^q, S^p \vee S^q)$, since $H_{p+q}(S^p \times S^q, S^p \vee S^q) \approx H_{p+q}(S^p \times S^q)$ from (6.13) and (6.15), and since (6.48) and (6.49) implies $c_1(S^p \vee S^q) = e$, we have that c_1 and, because homotopies induce isomorphisms in homology, that c and d map the natural generator of $H_{p+q}(S^p \times S^q)$ into $(-1)^p\tau[\alpha,\beta]$, i.e

(6.52) $\quad d(\iota) \ = \ (-1)^p\tau[\alpha,\beta]$

where ι is the natural generator of $H_{p+q}(S^p \times S^q)$. (6.52) together with the following proposition will prove the theorem.

(6.53) <u>Proposition</u>. $d(\iota_p \otimes \iota_q) = \tau\alpha * \tau\beta - (-1)^{pq}\tau\beta * \tau\alpha$,
where ι_p, ι_q are the natural generators of $H_p(S^p)$, $H_q(S^q)$.

<u>Proof</u>. It is immediate from (6.47) that d may be written as the following composition:

$$m \circ (m \times m) \circ (1 \times 1 \times \gamma \times \gamma) \circ (\alpha' \times \beta' \times \alpha' \times \beta') \circ (1 \times \lambda \times 1) \circ (\delta_1 \times \delta_2) : S^p \times S^q \to \Omega_X ,$$

where δ_1, δ_2 are the diagonal maps $\delta_1 : S^p \to S^p \times S^p$, and $\delta_2 : S^q \to S^q \times S^q$,

λ is the permutation map $\lambda : S^p \times S^q \to S^q \times S^p$ defined by (6.28).

γ is the inversion $\gamma : \Omega_X \to \Omega_X$ defined in (4.20), and

m is the multiplication $m : \Omega_X \times \Omega_X \to \Omega_X$ defined by (4.9).

We let ι_p be the natural generator of $H_p(S^p)$ and ι_q be the natural generator of $H_q(S^q)$, so that by definition $\alpha'(\iota_p) = \tau\alpha$ and $\beta'(\iota_q) = \tau\beta$. Again we let e be the generator of $H_o(x_o)$.

We want to determine $d(\iota_p \otimes \iota_q)$.

Since δ_1 and δ_2 are diagonal maps, we have

$$\delta_1(\iota_p) = \iota_p \otimes e + e \otimes \iota_p ,$$

$$\delta_2(\iota_q) = \iota_q \otimes e + e \otimes \iota_q ,$$

and

$$(6.54) \qquad \delta_1 \times \delta_2(\iota_p \otimes \iota_q).$$
$$= \iota_p \otimes e \otimes \iota_q \otimes e + \iota_p \otimes e \otimes e \otimes \iota_q + e \otimes \iota_p \otimes \iota_q \otimes e + e \otimes \iota_p \otimes e \otimes \iota_q .$$

From (6.24) , $\lambda(\iota_p \otimes \iota_q) = (-1)^{pq}\iota_q \otimes \iota_p$,

$$\lambda(\iota_p \otimes e) = e \otimes \iota_p ,$$

$$\lambda(e \otimes \iota_q) = \iota_q \otimes e ,$$

so that applying $1 \times \lambda \times 1$, (6.54) becomes

(6.55) $\quad \iota_p \otimes \iota_q \otimes e \otimes e + \iota_p \otimes e \otimes e \otimes \iota_q + e \otimes (-1)^{pq}\iota_q \otimes \iota_p \otimes e$
$\qquad + e \otimes e \otimes \iota_p \otimes \iota_q .$

Now

(6.56) $\quad \alpha' \times \beta' \times \alpha' \times \beta' \; (1 \times \lambda \times 1)(\delta_1 \times \delta_2)) \; (\iota_p \otimes \iota_q)$

$= \alpha'(\iota_p) \otimes \beta'(\iota_q) \otimes e \otimes e + \alpha'(\iota_p) \otimes e \otimes e \otimes \beta'(\iota_q)$
$\qquad + e \otimes \beta'((-1)^{pq}\iota_q) \otimes \alpha'(\iota_p) \otimes e + e \otimes e \otimes \alpha'(\iota_p) \otimes \beta'(\iota_q),$

and, if we consider the effect of $1 \times 1 \times \gamma \times \gamma$, (6.56) becomes

(6.57) $\quad \alpha'(\iota_p) \otimes \beta'(\iota_q) \otimes e \otimes e + \alpha'(\iota_p) \otimes e \otimes e \otimes \gamma(\beta'(\iota_q))$
$\qquad + e \otimes (-1)^{pq}\beta'(\iota_q) \otimes \gamma(\alpha'((\iota_p)) \otimes e + e \otimes e \otimes \gamma(\alpha'(\iota_p)) \otimes \gamma(\beta'(\iota_q)).$

Applying $m \times m$, (6.57) becomes

(6.58) $\quad \alpha'(\iota_p) * \beta'(\iota_q) \otimes e * e + \alpha'(\iota_p) * e \otimes e * \gamma(\beta'(\iota_q))$
$\qquad + e * (-1)^{pq}\beta'(\iota_q) \otimes \gamma(\alpha'(\iota_p)) * e + e * e \otimes \gamma(\alpha'(\iota_p)) * \gamma(\beta'(\iota_q)),$

so that by applying m , we have

(6.59) $\quad d(\iota_p \otimes \iota_q) = (\alpha'(\iota_p) * \beta'(\iota_q)) * (e * e) + (\alpha'(\iota_p) * e) * (e * \gamma(\beta'(\iota_q))$
$\qquad + (e * (-1)^{pq}\beta'(\iota_q)) * (\gamma(\alpha'(\iota_p)) * e) + (e * e) * (\gamma(\alpha'(\iota_p)) * \gamma(\beta'(\iota_q)).$

Since e is the unit of the Pontryagin algebra, (6.59) may be written as

(6.60) $\quad d(\iota_p \otimes \iota_q) = \alpha'(\iota_p) * \beta'(\iota_q) + \alpha'(\iota_p) * \gamma(\beta'(\iota_q)) + (-1)^{pq} \beta'(\iota_q) *$
$\qquad \gamma(\alpha'(\iota_p))$
$\qquad + \gamma(\alpha'(\iota_p)) * \gamma(\beta'(\iota_q)).$

We consider the composite

(6.61) $\quad m \circ (1 \times \gamma) \circ (\alpha' \times \alpha') \circ \delta_1 ,$

which is clearly identical to the composite

$$m \circ (1 \times \gamma) \circ \delta \circ \alpha' \quad ,$$

where δ is the diagonal map $\delta : \Omega_X \to \Omega_X \times \Omega_X$. Now the composite $m \circ (1 \times \gamma) \circ \delta : \Omega_X \to \Omega_X$ is null homotopic from (4.20), since $\gamma(f)$ is the homotopy inverse of f. Hence (6.61) is null homotopic.

Now

$$\begin{aligned} & m(1 \times \gamma(\alpha' \times \alpha'(\delta_1(\iota_p)))) \\ = {} & m(1 \times \gamma(\alpha' \times \alpha'(\iota_p \otimes e + e \otimes \iota_p) \\ = {} & m(1 \times \gamma(\alpha'(\iota_p) \otimes e + e \otimes \alpha'(\iota_p))) \\ = {} & m(\alpha'(\iota_p) \otimes e + e \otimes \gamma(\alpha'(\iota_p))) \\ = {} & \alpha'(\iota_p) * e + e * \gamma(\alpha'(\iota_p)) \\ = {} & \alpha'(\iota_p) + \gamma(\alpha'(\iota_p)) \end{aligned}$$

so that $\alpha'(\iota_p) = -\gamma(\alpha'(\iota_p))$. Similarly

$$\beta'(\iota_q) = -\gamma(\beta'(\iota_q)).$$

Substituting in (6.60) ,

$$\begin{aligned} d(\iota_p \otimes \iota_q) &= \alpha'(\iota_p) * \beta'(\iota_q) - \alpha'(\iota_p) * \beta'(\iota_q) - (-1)^{pq}\beta'(\iota_q) * \alpha'(\iota_p) \\ &\qquad + \alpha'(\iota_p) * \beta'(\iota_q) . \\ &= \alpha'(\iota_p) * \beta'(\iota_q) - (-1)^{pq}\beta'(\iota_q) * \alpha'(\iota_q) \end{aligned}$$

Substituting $\tau\alpha$ for $\alpha'(\iota_p)$ and $\tau\beta$ for $\beta'(\iota_q)$, we have

$$d(\iota_p \otimes \iota_q) = \tau\alpha * \tau\beta - (-1)^{pq}\tau\beta * \tau\alpha,$$

which completes the proof of (6.53).

Combining (6.53) and (6.52) we have

$$\tau[\alpha,\beta] = (-1)^p(\tau\alpha * \tau\beta - (-1)^{pq}\tau\beta * \tau\alpha)$$

which completes the proof of Theorem B.

Remark. We note that this second proof has precisely determined the sign to be $(-1)^p$, whereas the first proof left this sign open.

We used the concept of a universal example for a binary homotopy construction in the (first) proof of Theorem B. We also note it is partially because of a universal example for an n-ary homotopy construction that this theorem (and the Corollary (5.13) to Theorem A) is important in homotopy theory.

(6.62) Definition. We put $S = S_1 \vee S_2 \vee \ldots \vee S_n$, where the S_i, $1 \leq i \leq n$, are oriented spheres of dimension k_i with a point x_o in common, and let $\iota_i \in \pi_{k_i}(S,x_o)$ be the inclusion $(S_i,x_o) \to (S,x_o)$.

Blakers and Massey [2] have shown that S and the ι_i, $1 \leq i \leq n$ is a universal example for an n-ary homotopy constructions from dimensions $k_1,\ldots,k_n$ to any other dimension ℓ; such constructions being in 1 : 1 correspondence with the elements of $\pi_\ell(S)$. Clearly, to prove general theorems about such constructions, we often need only consider the universal example. Hence Hilton [9] showed the importance of Theorem A, Corollary (5.13), and Theorem B by using them to calculate the homotopy groups of S.

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